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Applications of the Weyl-Wigner formalism to noncommutative geometry

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Why noncommutative geometry?

It is well known that the classical formulation for the dynamics of a physical system is based on the notions of functions on a differentiable manifold, of diffeomorphism group actions and of vector field.

Pure states are represented by points of a manifold, observables by real functions on it. If this manifold is given both a topological and a differentiable structure, then time evolution is represented by a one parameter group of diffeomorphisms on it, whose infinitesimal generator is a vector field.

This formalization originated from the analysis of the concepts of position and velocity of a body, following the Newtonian's assumptions. Classical mechanics developed its own language. This language is differential geometry.

That this geometry were a branch of physics, quoting Einstein's opinion, is perfectly clear looking at the way classical general relativity and classical gauge theories describe gravitational and electromagnetic interactions.

Quantum formalization for the dynamics of a physical system is profoundly different. Pure states are represented by rays of a separable Hilbert space; observables are represented by self-adjoint operators on this space; time evolution is represented by unitary transformations on the set of states.

In his book on the principle of quantum mechanics [12], Dirac wrote that one of the dominant features of this scheme is that observables appear in it as quantities which do not obey the commutative law of multiplication. Moreover, noncommutativity among observables is exactly the way uncertainty relations, whose appearance is one among the most important differences between classical and quantum physics, are introduced in the formalism.

The very first example of uncertainty relations is that related to position and momentum observables for a quantum dynamics of point particles. Its mathematical formulation is based on the definition of canonical commutation relations:

$$[\hat{q}^a, \hat{p}_b] = i\hbar\delta^a_b$$

These relations introduce a correlation among the dispersions of the statistical distributions of measured values of positions and momenta. In the paper [13] Dirac was led to consider the possibility to describe quantum physics on the phase space carrying a classical dynamics of point particles. The phase space would have been recovered as the continuous spectra of a set of fundamental quantum observables. He introduced the notion of quantum algebra of functions, and of quantum analogue of classical derivations, calling them quantum differentiations. Above all, he was aware that the uncertainty relations drive to the impossibility of an infinitely precise localization of points on this phase space.

This example can be considered as the first noncommutative space. The impossibility of such a perfect localization shows that the "geometry" of this space should be considered to have lost the concept of point. "Pointless geometry" was exactly the name von Neumann [47] gave to the mathematical studies originated by the analysis of the quantum formalism.

He started introducing the concept of rings of operators (nowadays called von Neumann algebras) as a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space[1]. In this context, for the first time he related a topological

[1] A von Neumann algebra is a subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed under the involution

condition to an algebraic one. His primary motivation came from the hope to characterize quantum mechanical systems by algebraic conditions on the observables, rather then topological analysis on the set of states. This program would have been developed, and to some extent realised, in algebraic quantum field theory, and in the study of quantum statistical mechanics of infinite systems.

These research perspectives went on with the work of Gelfand who combined studies of operator algebras with the theory of Banach spaces. He introduced the notion of a Banach algebra in which multiplication is separately continuous in the norm topology, and formalised an intrinsic spectral theory. Then Gelfand and Neumark defined what is now called a C^*-algebra, and proved the basic theorem that each noncommutative C^*-algebra is isomorphic to a norm closed $*$-subalgebra of bounded operators on a Hilbert space. This theorem is now called GNS: its constructive proof was developed by Segal, who established a connection between states and representations. In [38] there is a beautiful account of an algebraic description of quantum mechanics.

These historical notes[2] are just to show that quantum physics required the development of a specific language. This language is noncommutative geometry, and noncommutative topology.

This formalism is meant to give primary importance to the notion of space (algebra) of observables, and to consider the state vector as a derived object. The notion of pure states replaces that of points, while derivations of the algebra replace vector fields. The aim of noncommutative geometry is to reformulate the geometry of a manifold in terms of features of the abelian algebra of functions defined on it (there exists a version of GNS theorem suited for commutative C^*-algebras), and then to generalize the corresponding results of differential geometry to the case of noncommutative algebras.

In the last years this program was evolved expecially by Connes, who extended the notion of exterior calculus and of de Rham cohomology to the noncommutative case. This, together with the theory of Hilbert modules, generalising the notion of fiber bundles, enables to study gauge theory in the noncommutative setting. Moreover, if classical general relativity is beautifully formalised using classical, say commutative, differential geometry, noncommutative geometry appears, in the current research activity, as the natural language to formalise quantum gravity, and, more generally, a unified quantum description of fundamental interactions.

Why Weyl-Wigner formalism?

After introducing the notions of quantum conditions (nowadays called commutation relations), Dirac went on writing:

> ...The problem of finding quantum conditions is not of such a general character...It is instead a special problem which presents itself with each particular dynamical system one is called upon to study.

$\hat{A} \to \hat{A}^\dagger$, and sequentially complete in the weak operator topology. This topology can be defined by its notion of convergence. A sequence $\{\hat{A}_n\}$ of bounded operators weakly converges to $\hat{A}$ when $\langle \psi \mid \hat{A}_n \mid \psi \rangle \to \langle \psi \mid \hat{A} \mid \psi \rangle$ for state vectors $\mid \psi \rangle$ in the Hilbert space $\mathcal{H}$. This type of convergence is partly motivated by quantum mechanics, in which $\langle \psi \mid \hat{A} \mid \psi \rangle$ represents the expectation value of the observable represented by $\hat{A}$ when the system is in the state represented by $\mid \psi \rangle$, provided that $\hat{A}$ is selfadjoint and $\langle \psi \mid \psi \rangle = 1$.

[2] A more refined and rigorous account of these topics is in [20].

> There is a fairly general method of obtaining quantum conditions, applicable to a very large class of dynamical systems. This is the method of *classical analogy*... Those dynamical systems to which this method is not applicable must be treated individually and special considerations used in each case.
>
> The value of classical analogy in the development of quantum mechanics depends on the fact that classical mechanics provides a valid description of dynamical systems under certain conditions, when the particles and bodies composing the system are sufficiently massive for the disturbance accompanying an observation to be negligible. Classical mechanics must therefore be a limiting case of quantum mechanics...

His suggestion to consider a classical analogy is nowadays referred to as a principle of correspondence. It is naively said that a quantum system should be formalised in such a way that, in the classical limit, formally obtained by letting $\hbar \to 0$, the corresponding classical dynamics could be recovered. The Weyl-Wigner formalism is a setting convenient to analyse the relations between the classical and the quantum formalism, and to put in a more rigorous form the problem of quantizing a classical dynamics, and of recovering the classical limit of a quantum dynamics.

and this dissertation...

The theme of this dissertation is to use the Weyl-Wigner formalism to study relations between quantum and classical physics, or, that is the same, to study relations between quantum and classical geometry.

In chapter 1 the Weyl-Wigner formalism is introduced, starting from a group theoretical interpretation of the cartesian phase space as a carrier space for a classical dynamics. The notion of Weyl system provides a more satisfactory account for the definition of the quantum conditions, i.e. the commutation relations for a certain class of quantum systems, and becomes a way to analyse the formulation of the principle of classical analogy.

Then the Weyl-Wigner map is introduced. It is a bijection between a set of operators on a Hilbert space, and a set of functions on a vector space. This map enables to write the noncommutative product among quantum observables as a noncommutative product among functions on this phase space. This is the Moyal product, and it explicitly depends on $\hbar$, reducing to the standard pointwise product in the limit $\hbar \to 0$. In this chapter it is described how this formalism makes it possible to study both the problem of quantizing certain classical systems, and the problem of formalizing the quantum evolution in the space of functions defined on the same phase space where classical observables are represented. In this setting, a classical limit procedure is written in a more rigorous form.

In chapter 2 this formalism is extended to the case of a quantum system corresponding to a classical dynamics whose phase space is the cotangent bundle of a compact, simple, Lie group, seen as a configuration space. Quantum conditions among fundamental quantum observables are here mutuated by the Lie algebra structure of the group. The novelty of this approach is that a Weyl-Wigner isomorphism is now realised between operators on a Hilbert space and functions which are defined not on the "classical" phase space, namely that cotangent bundle of the Lie group, but on the product of the group manifold, the configuration space, with a discrete space. This can be seen as a sort of quantum phase space. It explicitly depends on the global topological properties of the group, and on its nonabelianess. These aspects play a role to give the results of the specific harmonic analysis performed to define the isomorphism.

Nevertheless, a different isomorphism between operators on a Hilbert space and functions on the classical phase space can be written. This eventually enables to study noncommutative algebras of functions defined on the cotangent bundle of a compact simple Lie group.

In chapter 3 the machinery developed in the first part is fully used in the specific example of defining a new fuzzy space, the fuzzy disc.

A fuzzy space is a sequence of nonabelian algebras, more properly finite rank matrix algebras, approximating as "quantum metric spaces", the commutative algebra of functions on continuous manifolds on which field theory models are defined. This approximation is seen to act as a regulator for ultraviolet divergences in a class of field theory obtained via a canonical quantization of classical fields.

The chapter opens with a description of what is the fuzzy sphere (seen as a prototype of a fuzzy space) to describe what is the meaning of this approximation, and what is the meaning of the convergence of this sequence of nonabelian algebras towards an abelian one. Then the fuzzy disc is introduced, as the first example of a fuzzy approximation of a continuous space having a boundary. It is developed starting from the analysis of the noncommutative plane obtained via a standard Weyl-Wigner isomorphism. The stress is put on the way a sequence of finite rank matrix algebras is obtained, and the way the introduction of derivatives and a "fuzzy" Laplacian operator leads to the concept of "fuzzy Bessel functions". This notion extends that of fuzzy spherical harmonics introduced in the case of the sphere. On the fuzzy Bessels it is based the procedure by which the algebra of functions on a disc can be given a fuzzy version. Moreover, this approximation is seen to heal the ultraviolet divergences already present in noninteracting field theories on a disc.

At the end, some appendices recollect concepts used in the main text. The first is devoted to briefly introduce some of the algebraic concepts mentioned in the text. The second explains the meaning of Fourier symplectic transform, used to perform the harmonic analysis of the translation group. The third defines what is a system of generalised coherent states, which is a unifying scheme for the realization of the various kinds of Weyl-Wigner isomorphisms developed in the text. The last appendix recollects the calculations performed to obtain the form of the nonabelian product among functions defined on the quantum cotangent space defined in chapter 2.

Contents

Chapter 1

An introduction to the Weyl-Wigner formalism

There is a profound difference between the classical and the quantum formulation for the dynamics of a physical system. This difference, together with the hypothesis - a principle of correspondence - that the classical formalism were to be recovered as a limiting case of the quantum one, originate the problem of studying this comparison in a convenient setting.

The Weyl-Wigner formalism is a setting convenient for the analysis of some aspects of this problem. It shows a procedure in quantizing certain classical dynamics, and enables to give, for a class of quantum dynamics, a more consistent meaning to the formulation of the so called classical limit, often naively considered as a formal manipulation obtained by letting '$\hbar \to 0$'.

This formalism is based on an application, the Weyl map, that transforms functions defined on a real, even dimensional, vector space, into operators on a separable Hilbert space:

$$\hat{\Omega} : \mathcal{F}(S) \mapsto Op(\mathcal{H})$$

The Weyl map is invertible, and its inverse is called Wigner map. They are introduced via an explicit use of the fundamental constant $\hbar$, and of a symplectic 2-form on the vector space, so that this vector space can be identified with a phase space carrying a classical dynamics. This bijection can then be seen as a kind of map between the space of classical observables and the space of quantum observables.

These two spaces are very different. The composition rule among classical observables is abelian, while the product rule among quantum observables is non abelian. Weyl-Wigner map enables to translate the noncommutative product in the space of operators into a noncomutative product in the set of functions on the phase space carrying a classical dynamics. This means that the quantum algebra can be represented in terms of functions on the classical phase space, where a "$*$"-product is introduced:

$$f * g = \hat{\Omega}^{-1}\left(\hat{\Omega}(f) \cdot \hat{\Omega}(g)\right)$$

This is also called a Moyal product, and it is a deformation of the standard abelian pointwise product. Here deformation means that it explicitly depends on $\hbar$, and reduces to the pointwise product in the limit for $\hbar \rightarrow 0$, where Planck's constant is now seen as a parameter. The antisymmetrization of this product gives a Moyal bracket:

$$\{f, g\}_M = \frac{i}{\hbar}(f * g - g * f)$$

The same way the Moyal product represents the composition law among operators, this application is the definition, in the space of functions, of the notion of commutator of two operators. It is bilinear, skewsymmetric, and satisfies both the Jacobi identity and the Leibnitz rule. It is a deformation, in $\hbar$, of the Poisson Bracket of two functions. This means that this formalism recovers a more rigorous version of the analogy Dirac expressed, between the commutators (among quantum observables) and Poisson Brackets (among classical observables).

The first part of this chapter describes the notion of Weyl system, which is the building block in the construction of the Weyl and Wigner maps. The emphasys is given to the interpretation of a vector phase space as the manifold carrying a realization of the translation group, and of the Weyl system as a unitary projective representation associated with this realization. The phase factors are related to the symplectic structure defined on the group. In the analysis of the covariance properties of this representation for the action of the symplectic group, it is possible to find a procedure of quantization for some classical dynamics.

Then the Weyl and Wigner maps are presented, and the Moyal product is introduced, to describe a setting where the quantum evolution can be written in terms of equations on functions on the phase space. In this setting the role of $\hbar$ is made explicit, and recovering a classical limit is natural.

Afterwards two steps towards a generalization of the Weyl-Wigner isomorphism are elucidated. The first is the full analysis of the definition of this formalism in the case that classical dynamics can be written in terms of a generic, though still translationally invariant, symplectic form on the classical cartesian phase space. The second describes how the Weyl system's notion can be enlarged to the case where it is defined as a more general projective representation of the same translation group. The result is the introduction of the weighted Weyl maps, that clarify how also the ordering problems, usually mentioned in the formal quantization procedure, can be understood via this formalism.

The chapter ends with the proof that the standard Weyl formalism can be obtained without using the tools of harmonic analysis for the translation group, but studying the definition of a system of coherent states (the notion of generalized coherent states is introduced in the Appendix) for the Heisenberg-Weyl-Wigner group. It also shows one of the reason why, in this harmonic analysis, the concept of symplectic Fourier transform has been used. Even this concept is introduced in Appendix.

1.1 Weyl systems

1.1.1 From classical mechanics to quantum conditions

Dynamical evolution of a physical system can be represented as a set of transformations, parametrised by time, in the space of states of the system into itself [25]:

$$\mathcal{D}_t \,:\, \mathcal{S} \mapsto \mathcal{S}$$

These transformations represent the half-line as a one parameter semigroup $\left(\mathbb{R}_0^+, +\right)$, because they satisfy a composition rule, for every positive value of time parameter t:

$$\mathcal{D}_0 \,=\, \mathbf{1}$$

$$\mathcal{D}_{t'+t''} \,=\, \mathcal{D}_{t'} \,\circ\, \mathcal{D}_{t''} \,=\, \mathcal{D}_{t''} \,\circ\, \mathcal{D}_{t'}$$

If these $\mathcal{D}_t$ are bijective maps, then it is defined:

$$\mathcal{D}_t^{-1} \,\equiv\, \mathcal{D}_{-t}$$

and time evolution is seen as a one parameter group $\left(\mathbb{R}, +\right)$ of transformations of $\mathcal{S}$ into itself.

This section is devoted to the description of some aspects of the Poisson, and of the symplectic formulation of classical dynamical systems. The well established topics reviewed here[1] are meant to be steps of a path driving to the definition of the so called *quantum conditions*.

In classical mechanics [1] the set of pure states is formalised as a manifold $\mathcal{L}$ and observables are represented by real functions on this manifold. With respect to the topological and to the differentiable structure of $\mathcal{L}$, the evolution $\mathcal{D}_t$ of a classical dynamical system can be seen as a one parameter group of diffeomorphisms of $\mathcal{L}$, whose infinitesimal generator is a (complete) vector field. The integral curves of this vector field represent the evolution of pure states, while the evolution of observables can be written as

$$f_t \,=\, f \circ \phi_t$$

or, infinitesimally, as a solution of:

$$\frac{df}{dt} \,=\, X \cdot f \,=\, L_X f \tag{1.1}$$

(f is the function representing the observable, X is the vector field generating the dynamics, and $L_X f$ is the Lie derivative of f along X, i.e. the directional derivative of f along the integral curves of X.)

Given a manifold $\mathcal{L}$, a Poisson bracket is a map that associates a function on $\mathcal{L}$ to a pair of such functions:

$$\mathcal{F}\left(\mathcal{L}\right) \,\times\, \mathcal{F}\left(\mathcal{L}\right) \,\mapsto\, \mathcal{F}\left(\mathcal{L}\right)$$

which is bilinear, skewsymmetric, and satisfies:

[1]For the concepts of manifold analysis and calculus, assumed in these pages, an excellent textbook is [1].

- Jacobi identity, for every triple of functions:

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0$$

- Leibnitz rule, for every triple of functions:

$$\{fg,h\} = f\{g,h\} + \{f,h\}g$$

with respect to the standard pointwise product among elements in $\mathcal{F}(\mathcal{L})$. The Poisson bracket is regular if the requirement $\{f,g\} = 0 \ \forall f \in \mathcal{F}(\mathcal{L})$ implies that g is constant.

A map $\phi : \mathcal{L} \mapsto \mathcal{L}$ is called canonical if it preserves the Poisson tensor:

$$\{f,g\} \circ \phi \equiv \{f \circ \phi, g \circ \phi\}$$

$\forall f,g \in \mathcal{F}(\mathcal{L})$. A classical dynamics has a canonical formulation if time evolution is represented by a one parameter group of canonical (with respect to a given Poisson structure) transformations.

This formulation can be given an infinitesimal version. Since Poisson bracket satisfies the Leibnitz rule, for a fixed $H \in \mathcal{F}(\mathcal{L})$ the map:

$$\{\cdot, H\} : \mathcal{F}(\mathcal{L}) \mapsto \mathcal{F}(\mathcal{L})$$

is a derivation in the abelian algebra $\mathcal{F}(\mathcal{L})$. In the theory of differentiable manifolds, it is possible to prove that each smooth derivation on $\mathcal{F}(\mathcal{L})$ can be given the form of a vector field. To the Poisson derivation defined by H a vector field X_H can be associated, such that:

$$\{f,H\} = L_{X_H} f \tag{1.2}$$

X_H is called Hamiltonian vector field of Hamiltonian function H with respect to the given Poisson bracket. The components of this Hamiltonian vector field are, in a local coordinate chart $\{\xi^a\}$:

$$\dot{\xi}^a = \frac{d\xi^a}{dt} = \{\xi^a, H\}$$

so that the Poisson bracket of two functions can be written as:

$$\{f,g\} = X_g \cdot f = \{f,\xi^b\}\frac{\partial g}{\partial \xi^b} = \frac{\partial f}{\partial \xi^a}\{\xi^a,\xi^b\}\frac{\partial g}{\partial \xi^b}$$

Poisson bracket is defined by the components:

$$\Lambda_{ab} \equiv \{\xi^a,\xi^b\} \tag{1.3}$$

of a skewsymmetric twice contravariant tensor:

$$\Lambda \equiv \Lambda^{ab}\frac{\partial}{\partial \xi^a} \otimes \frac{\partial}{\partial \xi^b} \tag{1.4}$$

such that[2]:

$$\{f,g\} = \Lambda(df,dg) \tag{1.5}$$

[2]Jacobi identity is written as:

$$\Lambda^{sk}\left(\frac{\partial}{\partial \xi^s}\Lambda^{ab}\right) + \Lambda^{sa}\left(\frac{\partial}{\partial \xi^s}\Lambda^{bk}\right) + \Lambda^{sb}\left(\frac{\partial}{\partial \xi^s}\Lambda^{ka}\right) = 0$$

A one parameter group ϕ_s of canonical transformations on $\mathcal{L}$ is generated by an infinitesimal canonical vector field X:

$$L_X \Lambda = 0 \Leftrightarrow L_X\{f,g\} = \{L_X f, g\} + \{f, L_X g\} \tag{1.6}$$

An Hamiltonian vector field is a canonical vector field, so time evolution generated by an Hamiltonian vector field is canonical, i.e. it preserves the Poisson tensor Λ. The infinitesimal form of the evolution equations for a dynamical system in the Poisson formalism is:

$$\frac{df}{dt} = \{f, H\} \tag{1.7}$$

It is interesting to note that the set of Hamiltonian vector field does not exhaust the set of canonical vector field. There are canonical vector fields which are not Hamiltonian [8]. This may happen either because the Poisson bracket is degenerate, or because the carrier space manifold is not simply connected.

If Λ is a regular Poisson tensor whose matrix elements are Λ^{ab}, that is globally invertible, then it is possible to introduce a skewsymmetric twice covariant tensor on $\mathcal{L}$. In this coordinate chart:

$$\omega = \omega_{ab}\, d\xi^a \otimes d\xi^b$$

where

$$\Lambda^{ab}\, \omega_{bc} \equiv -\delta^a_c \tag{1.8}$$

This tensor is nondegenerate, and it can be proved that [8] the requirement that Jacobi identity is verified by Λ is equivalent to the closedness condition for this form: $d\omega = 0$. The pair $(\mathcal{L}, \omega)$, with $\mathcal{L}$ a differentiable manifold and ω a closed, nondegenerate, skewsymmetric covariant 2-form is called a symplectic manifold, and ω is a symplectic structure.

A vector field X is called locally Hamiltonian if it is the infinitesimal generator of symplectic transformations, those preserving the symplectic tensor:

$$L_X \omega = 0 \tag{1.9}$$

The symplectic vector fields are canonical with respect to the associated Poisson tensor (1.8):

$$L_X \Lambda = 0 \Leftrightarrow L_X \omega = 0 \tag{1.10}$$

From Cartan's identity:

$$\begin{aligned} L_X \omega &= 0 \rightarrow \\ i_X\, d\omega + d\, i_X \omega &= 0 \rightarrow \\ d\, i_X \omega &= 0 \end{aligned} \tag{1.11}$$

Globally Hamiltonian vector fields are those fields for which the 1-form $i_X \omega$ is not only closed, but exact:

$$i_{X_H} \omega = dH \tag{1.12}$$

A classical dynamics has an Hamiltonian formulation if it is infinitesimally represented by an Hamiltonian vector field.

For example, if $\mathcal{Q}$ is the configuration space for the evolution of a conservative Newtonian system, whose equations of motions are written as (q^a are position coordinates, v^a are velocities coordinates, this evolution being formalised on $T\mathcal{Q}$, the tangent bundle of $\mathcal{Q}$):

$$\begin{aligned} \dot{v}^a &= F^a\left(q^i, v^j\right) \\ \dot{q}^a &= v^a \end{aligned} \tag{1.13}$$

then, on the cotangent bundle $\mathcal{L} = T^*\mathcal{Q}$ (where q^i are still coordinates on $\mathcal{Q}$, the basis of the bundle, labelling position observables, while p_a are local coordinates on the fibers, labelling momenta), this evolution is symplectic via the definition of[3]:

$$\begin{aligned} \tilde{\Lambda}^a_b &\equiv \{q^a, p_b\} = \delta^a_b \\ \tilde{\Lambda} &= \frac{\partial}{\partial q^a} \wedge \frac{\partial}{\partial p_a} \\ \tilde{\omega} &= dq^a \wedge dp_a \end{aligned} \tag{1.14}$$

because equations of motions (1.13) are written as[4]:

$$\begin{aligned} \dot{q}^a = p^a &= \{q^a, H\} = \frac{\partial H}{\partial p_a} \\ \dot{p}_a = -\frac{\partial U}{\partial q^a} &= \{p_a, H\} = -\frac{\partial H}{\partial q^a} \end{aligned} \tag{1.15}$$

with $H = \frac{1}{2}p^a p_a + U\left(q^a\right)$ where U is the potential energy whose gradient is the force field F: $F_a = -\frac{\partial U}{\partial q^a}$.

It is possible to consider a symplectic manifold $(\mathcal{L}, \omega)$, such that it is an homogeneous space for a transitive and free action of the group of translations. If the system of coordinates (q^a, p_b) is global, and canonically adapted to this action, then the coordinate functions are seen to generate the Hamiltonian vector fields that represents this group of translations:

$$\begin{aligned} q^a \to X_{(q^a)} &= -\frac{\partial}{\partial p_a} \\ p_a \to X_{(p_a)} &= \frac{\partial}{\partial q^a} \end{aligned} \tag{1.16}$$

1.1.2 Standard Weyl systems

Quantum formulation for the dynamics of a physical system is profoundly different. Pure state are represented by rays of a separable Hilbert space; observables are represented by self-adjoint operators on this space. Time evolution is represented by unitary transformations on this set of states.

The principle of correspondence suggests to postulate that, in quantum theory, the groups generated by the cartesian position and momentum coordinates

[3]This form of the symplectic structure is very important, and it is called 'canonical'. The theorem of Darboux [1] proves that, given a symplectic manifold $(\mathcal{L}, \omega)$, there exists a local coordinate transformation on $\mathcal{L}$ such that the simplectic structure locally acquires the canonical form.

[4]In these expressions, indices are raised and lowered, to keep track of labelling covariant or contravariant elements of the tensor algebra on $\mathcal{L}$, by the metrics that is implicitly assumed in the definition of the kinetic energy term in the Hamiltonian function.

of a system of particles are the same as the classical ones (1.16), i.e. they displace, respectively, in the momenta and the positions [43]. The aim of this section is to describe how the notion of *Weyl system* is able to fulfill this expectation, and to show how it can be used to set up a procedure of quantization for a set of specific classical dynamics.

The mathematical formulation of this basic postulate is the introduction of the *canonical commutation relations* among quantum observables of positions and momenta:

$$[\hat{q}^a, \hat{p}_b] = i\hbar \mathbf{1} \delta^a_b \tag{1.17}$$

These canonical commutation relations were introduced by Dirac to stress the formal analogy between the properties of the Poisson bracket among classical observables, and those of the commutator among quantum obsevables. Weyl [48] was the first to study the problems of a concrete realization of these operators.

Given a pair of self-adjoint operators satisfying the canonical commutation relations, it is possible to prove that they cannot be both bounded. Let $\hat{A}$ and $\hat{B}$ two bounded operators, whose commutator is a multiple of the identity:

$$\left[\hat{A}, \hat{B}\right] = c\mathbf{1}$$

Boundedness of them both would imply that:

$$\left[\hat{A}, \hat{B}^n\right] = cn\hat{B}^{n-1}$$

together with triangle inequality for the operator norm, this would led to:

$$cn \parallel \hat{B} \parallel^{n-1} = cn \parallel \hat{B}^{n-1} \parallel \leq 2 \parallel \hat{A} \parallel \parallel \hat{B} \parallel^n$$

so one would finally have, $\forall\, n$:

$$cn \leq 2 \parallel \hat{A} \parallel \parallel \hat{B} \parallel$$

This is a contradiction to the hypothesis of boundedness of both $\hat{A}$ and $\hat{B}$. This result is known as Wintner's theorem. One of its consequences is that the equality between l.h.s and r.h.s. of eq.(1.17) is strictly valid not on every element of the Hilbert space on which those obsevables are represented. This is usually referred to writing the canonical commutation relations as:

$$[\hat{q}^a, \hat{p}_b] \subset i\hbar \mathbf{1} \delta^a_b \tag{1.18}$$

Weyl suggested to look at them in a more general context. A *symplectic vector space* is a pair (L, ω) consisting of a real topological vector space L, equipped with a continuous antisymmetric, nondegenerate bilinear form ω on L. 'Nondegenerate' means that if $\omega(z, u) = 0\ \forall\, z \in L$, then $u = 0$: this is an example of a symplectic manifold. Let (L, ω) be a symplectic vector space[5]: a *Weyl system* is for (L, ω) is a map into the set of unitary operators on a separable Hilbert space $\mathcal{H}$:

$$\hat{D} : L \mapsto \mathcal{U}(\mathcal{H}) \tag{1.19}$$

such that:

[5] In this analysis, L will always be finite dimensional. The requirement that ω were nondegenerate forces then the space to be even-dimensional. An interesting analysis of Weyl systems on infinite dimensional vector spaces, suitable for a formalization of field theories is in [2]

- $\hat{D}$ is continuous in the strong operator topology,
- for each pair of vectors z and u in L:

$$\hat{D}(z+u) = e^{i\omega(z,u)/2\hbar}\hat{D}(z)\hat{D}(u) \tag{1.20}$$

This condition is equivalent to:

$$\hat{D}(z)\hat{D}(u) = e^{-i\omega(z,u)/\hbar}\hat{D}(u)\hat{D}(z) \tag{1.21}$$

If the vector space L is identified with the manifold representing the abelian Lie group of translations, then a Weyl system can be seen as a unitary, projective representation of the translation group. The phase factor of this representation is related to the symplectic form on L. Linearity of ω can be considered as the invariance of the this symplectic tensor with respect to the action of the translation group. This also suggests the reason why operators $\hat{D}$ are called *Displacement operators.*

On a one dimensional subspace of L, with α and β real scalars, the phase factors cancel out:

$$\hat{D}(\alpha z)\hat{D}(\beta z) = \hat{D}((\alpha+\beta)z) \tag{1.22}$$

this means that $\hat{D}(\alpha z)$ is a strongly continuous, one parameter group of unitary operators. The theorem of Stone says that it can be considered as the exponentiation of a self-adjoint operator:

$$\hat{D}(\alpha z) = e^{i\alpha G(z)/\hbar} \tag{1.23}$$

Moreover, it can be seen that, up to additive terms like $2\pi n\mathbf{1}$ in the identification of generators of a Weyl system, one has:

$$\hat{G}(\alpha z) = \alpha\hat{G}(z) \tag{1.24}$$

These generators have important properties. The defining relation (1.21) suggests that :

$$\hat{D}(\alpha z)\hat{D}(\beta u) = e^{-i\omega(\alpha z,\beta u)/\hbar}\hat{D}(\beta u)\hat{D}(\alpha z) \tag{1.25}$$

If they are written in terms of generators:

$$e^{i\alpha\hat{G}(z)/\hbar}e^{i\beta\hat{G}(u)/\hbar} = e^{i\alpha\beta\omega(z,u)/\hbar}e^{i\beta\hat{G}(u)/\hbar}e^{i\alpha\hat{G}(z)/\hbar} \tag{1.26}$$

This relation shows, once more, that in the Weyl approach every one dimensional subspace of L corresponds to a one parameter group of unitary operators. They will not suffer any of the problem of unbounded operators. It can be seen as a global version, in terms of bounded operators, of the canonical commutation rules.

If parameters α and β are considered as infinitesimal, one has

$$[\mathbf{1}+i\alpha\hat{G}(z)/\hbar+o(\alpha^2)][\mathbf{1}+i\beta\hat{G}(u)/\hbar+o(\beta^2)] =$$
$$\left(1-i\omega(z,u)\alpha\beta/\hbar+o\left((\alpha\beta)^2\right)\right)[\mathbf{1}+i\beta\hat{G}(u)/\hbar+o(\beta^2)]\cdot[\mathbf{1}+i\alpha\hat{G}(z)/\hbar+o(\alpha^2)]$$

and equating the coefficients of the first nonzero order in these infinitesimals, one obtains:

$$\left[\hat{G}(z),\hat{G}(u)\right] = i\hbar\omega(z,u)\mathbf{1} \tag{1.27}$$

This analysis is again performed as a formal manipulation: it could give no more than a hint to prove a rigorous result [2]. If $\hat{W}(z)$ defines a Weyl system for (L, ω), whose generators are selfadjoint $\hat{G}(z)$ then, for arbitrary vectors z and u in L, a dense domain for the products of the two generators, $Dom\left(\hat{G}(z)\hat{G}(u)\right) = Dom\left(\hat{G}(u)\hat{G}(z)\right)$, can be found and, for every element ϕ of the Hilbert space belonging to this domain, one has:

$$\left[\hat{G}(z), \hat{G}(u)\right]\phi = i\hbar\omega(z, u)\phi \tag{1.28}$$

This is the exact form of eq.(1.18).

So canonical commutation rules can be seen as the infinitesimal version of (1.26). They involve the selfadjoint generators of the group of displacements, recovered as observables. Noncommutativity among quantum observables is introduced via the symplectic form, and "measured" by $\hbar$.

The presence of the symplectic structure in the formalization of the *quantum conditions* is a first appearance of the principle of *analogy*. The quantum formalism is developed on the geometric structures on which classical formalism is based.

1.1.3 The Schrödinger representation

So far a Weyl system has been defined, and its properties have been deduced at a formal level. Now it will be explicitly realized.

Let M be a finite-dimensional real vector space, M^* its dual, the space of linear functions on M. Their direct sum define an even dimensional real vector space $L \equiv M \oplus M^*$, whose vectors z are of the form $z = x \oplus \lambda$. A symplectic form is given by $\tilde{\omega}(z, z') \equiv \lambda'(x) - \lambda(x')$. If M is considered as a manifold, then L is the cotangent bundle of M: $L \simeq T^*M$, and vectors z can be written in terms of a global coordinate chart as $z = (q^a, p_b)$. This notation, with indices a, b ranging from 1 to the dimension of M, makes it explicit that, in the identification of L with T^*M, q^a coordinates label elements of the basis M, while p_b coordinates label elements of the fiber M^*. In this coordinate chart, the symplectic tensor assumes the canonical, Darboux form $\tilde{\omega} = dq^a \wedge dp_a$, so that eq.(1.27) gives the canonical commutation relations (1.17) for the generators of a Weyl system.

Let $\mathcal{H} = \mathcal{L}^2(M, dx)$ be the Hilbert space of square integrable functions on M with respect to the translationally invariant Lebesgue measure. A pair of one parameter groups of unitary operators can be defined:

$$\begin{aligned} \left(\hat{U}(q)\psi\right)(x) &= \psi(x+q) \\ \left(\hat{V}(p)\psi\right)(x) &= e^{i\langle p, x\rangle/\hbar}\psi(x) \end{aligned} \tag{1.29}$$

Here q is a vector of coordinates q^a, labelling a x element of M, while p is a covector of coordinates p_b labelling a λ element of M^*; $\langle p, x\rangle$ represents the action of the function λ on the element x. $\hat{U}(q)$ is a unitary faithful representation of the abelian group of translations $(M, +)$. Harmonic analysis for this group says that its dual M^* is isomorphic to M, and $\hat{V}(p)$ can be considered as a unitary faithful representation of $(M^*, +)$. These representations are not mutually commuting:

$$\hat{U}(q)\hat{V}(p) = e^{i\tilde{\omega}((q,0),(0,p))/\hbar}\hat{V}(p)\hat{U}(q) \tag{1.30}$$

In this expression $\tilde{\omega}((q,0),(0,p)) = qp$ indicates the image of the 2-form on the pair of elements in L whose components are $(q,0)$ and $(0,p)$. Let:

$$\hat{D}(q,p) = \hat{U}^{\dagger}(q)\,\hat{V}(p)\,e^{i\langle p,q\rangle/2\hbar} \tag{1.31}$$

define a set of operators. They are unitary operators, in correspondence with points of the space $L = T^{*}M$, which satisfy the required properties (1.20). These operators are a realization of a Weyl system for $(L,\tilde{\omega})$. Their image on a function ψ in $\mathcal{H}$, in the chosen realization, is:

$$\left(\hat{D}(q,p)\,\psi\right)(x) = e^{-\frac{i}{2\hbar}\langle p,q\rangle}e^{\frac{i}{\hbar}\langle p,x\rangle}\psi(x-q) \tag{1.32}$$

Now, coming back to the pair of one parameter groups of unitary operators $\hat{U}(q)$ and $\hat{V}(p)$, one can consider the value of the coordinates q and p as infinitesimal parameters, to obtain an explicit form of their generators:

$$\begin{aligned} \hat{U}(q) &= e^{iq^{a}\hat{P}_{a}/\hbar} \quad \left(\hat{P}_{a}\psi\right)(x) = -i\hbar\frac{d\psi}{dx^{a}} \\ \hat{V}(p) &= e^{ip_{b}\hat{Q}^{b}/\hbar} \quad \left(\hat{Q}^{b}\psi\right)(x) = x^{b}\psi(x) \end{aligned} \tag{1.33}$$

Commutation relations (1.28) become:

$$\left[\hat{Q}^{a},\hat{P}_{b}\right] = i\hbar\delta^{a}_{b}\mathbf{1} \tag{1.34}$$

These are exactly the standard $\hat{Q}$ and $\hat{P}$ operators in the space of square integrable functions on linear space, that are a formal, well known solution to the problem of realizing the canonical observables of position and momentum in point particle quantum mechanics. The Displacement operators acquire the form:

$$\hat{D}(q,p) = e^{i\left(p_{a}\hat{Q}^{a}-q^{b}\hat{P}_{b}\right)/\hbar} \tag{1.35}$$

Weyl approach stresses the group theoretical interpretation of the classical phase space as the manifold that represents the translation group. Operators related to the observables position and momentum are recovered as generators of a representation of this action. This is the solution to the initial problem of defining quantum displacements. Moreover, the phase factor that characterizes this representation reproduces the commutation relations among its generators.

This explicit realization of a Weyl system is called Schrödinger representation. It can be proved to be irreducible. A fundamental result is known as Von Neumann theorem [34]. If $\hat{U}(q)$ and $\hat{V}(p)$ are strongly continuous one parameter group of unitary operators on a separable Hilbert space $\mathcal{H}'$, satisfying, in an irreducible representation, the Weyl form of the commutation relations:

$$\hat{U}(q)\,\hat{V}(p) = e^{i\langle p,q\rangle/\hbar}\hat{V}(p)\,\hat{U}(q) \tag{1.36}$$

then there is an isometry:

$$S : \mathcal{H}' \mapsto \mathcal{H} = \mathcal{L}^{2}(M,dx)$$

such that, for $\psi \in \mathcal{H}$:

$$\left(S\hat{U}(q)\,S^{-1}\psi\right)(x) = \psi(x+q)$$

$$\left(S\hat{V}\left(p\right)S^{-1}\psi\right)(x) = e^{i\langle p,q\rangle/\hbar}\psi((x)$$

This isometry says that there is an equivalence among irreducible solutions of the Weyl commutation relations[6].

This property is very important in the study of the problem of quantizing a classical system. The notion of Weyl system is based, on the vector space structure of L, and the invariance of the symplectic form for the action of the translation group. It is natural to study the covariance properties of this map for transformations of L. In particular, such transformations will respect, in a first approach, both the linear structure of L and the symplectic structure $\tilde{\omega}$: they will be linear and symplectic. A transformation T belongs to the linear symplectic group $Sp\left(L,\tilde{\omega}\right)$ if:

$$\tilde{\omega}\left(Tz,Tu\right) = \tilde{\omega}\left(z,u\right)$$

Acting with such a transformation on L, one has:

$$\hat{D}\left(Tz+Tu\right) = \hat{D}\left(Tz\right)\hat{D}\left(Tu\right)e^{\frac{i}{2\hbar}\tilde{\omega}(Tz,Tu)} \tag{1.37}$$

by linearity:

$$\hat{D}\left(T\left(z+u\right)\right) = \hat{D}\left(Tz\right)\hat{D}\left(Tu\right)e^{-\frac{i}{2\hbar}\tilde{\omega}(z,u)} \tag{1.38}$$

it is possible to define:

$$\hat{D}\left(Tz\right) \equiv \hat{D}_T\left(z\right) \tag{1.39}$$

obtaining:

$$\hat{D}_T\left(z+u\right) = \hat{D}_T\left(z\right)\hat{D}_T\left(u\right)e^{-\frac{i}{2\hbar}\tilde{\omega}(z,u)} \tag{1.40}$$

This means that $\hat{D}_T$ is a new Weyl system for $(L,\tilde{\omega})$, so it is unitarily equivalent to $\hat{D}\left(z\right)$. This equivalence enables to associate an automorphism to this transformation T

$$\nu_T\ : \mathcal{U}\left(\mathcal{H}\right) \mapsto \mathcal{U}\left(\mathcal{H}\right)$$

by putting:

$$\hat{D}\left(Tz\right) \equiv \nu_T\left(\hat{D}\left(z\right)\right) \tag{1.41}$$

[6]In the Weyl approach, selfadjoint operators satisfying peculiar commutation relations are recovered as generators of suitable strongly continuous one-parameter groups of unitary operators. It is interesting to note that it is possible to define [34] the Hilbert space $\mathcal{L}^2\left(Q,d\mu\right)$ where Q is the Riemann surface of the $\sqrt{z}$, $(z = x+iy)$and $d\mu$ is the (local) Lebesgue measure. On this Hilbert space it can be proved to exist a domain $\mathcal{D}$ and a pair of operators:

$$\hat{A} \equiv -i\frac{\partial}{\partial x} \qquad \hat{B} \equiv x - i\frac{\partial}{\partial y}$$

which satisfy:

- $$\hat{A}\ ,\ \hat{B}\ : \mathcal{D} \mapsto \mathcal{D}$$
  $\mathcal{D}$ is a common domain of essential self-adjointness for the operators $\hat{A}$, $\hat{B}$
- For all $\phi \in \mathcal{D}$
  $$\hat{A}\hat{B}\phi - \hat{B}\hat{A}\phi = -i\phi$$

Nevertheless it can be proved that the unitary groups that these two operators generate do not satisfy the Weyl relations (1.30).

Since every automorphism for the group of unitary operators on a Hilbert space can be written as a conjugation by a unitary operator, this automorphism can be given the form:

$$\nu_T\left(\hat{D}(z)\right) = \hat{U}_T^{-1}\hat{D}(z)\,\hat{U}_T \tag{1.42}$$

where $\hat{U}_T$ is a unitary transformation[7].

This analysis also shows the reason why, in (1.33), the operator $\hat{Q}^a$ has a superscript index, while $\hat{P}_b$ has a subscript index. Given O, an automorphism of the linear space M, represented as a map $q'^a = O^a_b q^b$, it can be lift to a symplectic automorphism $\mathcal{O}$ of $(L, \tilde{\omega})$, defining a transformation on fibers: $p'_a \equiv \left(O^{-1}\right)^b_a p_b$.

Unitary equivalence expressed by (1.42) is written as:

$$\begin{aligned}\hat{U}^\dagger_{\mathcal{O}}\, e^{ip_a\hat{Q}^a/\hbar}\,\hat{U}_{\mathcal{O}} &= e^{i\left(O^{-1}\right)^a_b p_a\hat{Q}^b/\hbar} \\ \hat{U}^\dagger_{\mathcal{O}}\, e^{iq^a\hat{P}_a/\hbar}\,\hat{U}_{\mathcal{O}} &= e^{iO^a_b q^b\hat{P}_a/\hbar}\end{aligned} \tag{1.43}$$

and these two expressions say that $\mathcal{O}$ symplectic transformation on L acts on generators of the Weyl system, by linearity via the linear transformations O and O^{-1}:

$$\begin{aligned}\hat{P}_{(\mathcal{O})\,a} &= O^b_a\hat{P}_b \\ \hat{Q}^a_{(\mathcal{O})} &= \left(O^{-1}\right)^a_b\hat{Q}^b\end{aligned} \tag{1.44}$$

In particular, the transformation law for $\hat{P}$'s operators is contravariant with respect to that of $\hat{Q}$'s operators.

This equivariance of Weyl systems by the linear symplectic group is very interesting. One can consider a classical dynamics on L that admits an Hamiltonian description in terms of the symplectic structure in the canonical form and of a quadratic Hamiltonian function (1.15). Its time evolution is written as a one parameter group of linear symplectic map on L. Via this formalism, it is possible to associate, with this classical evolution, a one parameter group of unitary operators on a Hilbert space, that is a quantum evolution. This is a quantization procedure for certain classical dynamics, for example those of the free particle, and of the harmonic oscillator.

1.2 Weyl map

Using a realization of a Weyl system for a symplectic vector space as a map into the set of unitary operators on a Hilbert space, it is possible to define a map from a set of functions defined on this vector space, to a larger class of operators on the same Hilbert space. This application is called *Weyl map* [48]:

$$\hat{D}\,:(L,\tilde{\omega})\,\mapsto\,\mathcal{U}(\mathcal{H})$$

[7] It can be proved that the operator $\hat{U}_T$ is determined by the transformation T up to a phase. These phases cannot be totally eliminated, but can be at best reduced to a sign ambiguity. For such a pair of transformations T and T':

$$\hat{U}_T\hat{U}_{T'} = \pm\hat{U}_{TT'}$$

This situation can be expressed by saying that one is dealing with a representation of the $Mp(2)$, the metaplectyc group, which is a double covering of $Sp(2)$.

$$\hat{\Omega} : \mathcal{F}(L) \mapsto \mathcal{O}p(\mathcal{H})$$

In his book, Weyl considered the form of the unitary operators (1.31) in terms of the Hermitian generators. The expression becomes:

$$\hat{D}(q,p) = e^{i\left(p_a \hat{Q}^a - q^a \hat{P}_a\right)/\hbar} \tag{1.45}$$

His suggestion was to look at them as a sort of formal plane wave basis in a suitable space of operators. Following this idea, coefficients of an operator expansion are given by Fourier coefficients of a function on the plane. Since the geometrical structure underlying the whole construction is the symplectic form on L, then the Fourier transform is defined using this 2-form[8]: (the dimension of L is $2n$)

$$\tilde{f}(w) = \int \frac{dz}{(2\pi\hbar)^n} f(z)\, e^{-i\tilde{\omega}(z,w)/\hbar} \tag{1.46}$$

The formal definition of the Weyl map is:

$$\hat{\Omega}(f) = \hat{f} = \int \frac{dw}{(2\pi\hbar)^n} \tilde{f}(w)\, \hat{D}(w) \tag{1.47}$$

This integral should be understood in a distributional sense[9]: in terms of a generalized basis of the Hilbert space, it is possible to formally estimate, the trace of the operators

$$Tr\left[\hat{D}(z)\right] = (2\pi\hbar)^n \delta(z) \tag{1.49}$$

from which one has:

$$Tr\left[\hat{f}\right] = \frac{1}{(2\pi\hbar)^n} \int dz\, f(z) \tag{1.50}$$

The group properties of these $\hat{D}(z)$ operators, together with this distributional trace estimate, give:

$$Tr\left[\hat{D}(z)\,\hat{D}^\dagger(u)\right] = (2\pi\hbar)^n \delta(z-u) \tag{1.51}$$

This means that the $\hat{D}$ operators define a generalized resolution of the identity in the space of operators: then Weyl map can be inverted. The inverse is given by:

$$\tilde{f}(w) = Tr\left[\hat{f}\hat{D}^\dagger(w)\right]$$

$$f(z) = \int \frac{dw}{(2\pi\hbar)^n} e^{-i\tilde{\omega}(w,z)/\hbar} Tr\left[\hat{f}\hat{D}^\dagger(w)\right] \tag{1.52}$$

This is usually called *Wigner map.* Given an operator $\hat{A}$, the function $A(z)$ to which it is associated by the Wigner map is called *Weyl symbol.* One can see that Weyl-Wigner map takes the notion of Hermitian conjugation in the space

[8] In appendix A.2 there is a description of the symplectic Fourier transform.

[9] This means that, for a pair of vectors ψ and ψ' in the Hilbert space $\mathcal{H}$, one properly has:

$$\langle \psi \mid \hat{\Omega}(f) \mid \psi' \rangle \equiv \int \frac{dw}{(2\pi\hbar)^n} \tilde{f}(w) \langle \psi \mid \hat{D}(w) \mid \psi' \rangle \tag{1.48}$$

This definition will be implicitly used in the following, to evaluate the operators that the Weyl map associates to the coordinate functions.

of operators into that of complex conjugation in the space of symbols. If $A(z)$ is the symbol of $\hat{A}$, then:

$$\hat{\Omega}^{-1}\left(\hat{A}^{\dagger}\right) = A^{*}(z) \tag{1.53}$$

To study the properties of this application it is useful to consider a function f which is square-integrable on the plane, so that Plancherel theorem assures that Fourier transform is well defined. The action of the operator $\hat{f}$ in the Schrödinger representation (1.32) on $\mathcal{H} = \mathcal{L}^2\left(M \simeq \mathbb{R}^n, ds\right)$ is:

$$\begin{aligned} \left(\hat{f}\psi\right)(s) &= \int \frac{dz}{(2\pi\hbar)^n} \int \frac{dw}{(2\pi\hbar)^n} f(z)\, e^{-i\tilde{\omega}(z,w)/\hbar}\, e^{-\frac{i}{2\hbar}\langle k,x\rangle} e^{\frac{i}{\hbar}\langle k,s\rangle} \psi(s-x) \\ &= \frac{1}{(\pi\hbar)^n} \int dz\, f(z)\, e^{2i\langle (s-q),p\rangle/\hbar} \psi(2q-s) \end{aligned} \tag{1.54}$$

In these equations it has been used the notation: $z = q \oplus p$ and $w = x \oplus k$. Now the product:

$$\hat{W}(z) = 2^n e^{2i\left(p_a \hat{Q}^a - q^a \hat{P}_a\right)/\hbar} \hat{\mathcal{P}}$$

with $\left(\hat{\mathcal{P}}\psi\right)(s) = \psi(-s)$ the parity operator, defines a new set of Hermitian operators, a new resolution of the identity:

$$\begin{aligned} \hat{W}(z) &= 2^n\, \hat{D}(2z)\, \hat{\mathcal{P}} \\ \hat{W}(z) &= \hat{W}^{\dagger}(z) \\ Tr\left[\hat{W}(z)\, \hat{W}^{\dagger}(u)\right] &= (2\pi\hbar)^n\, \delta(z-u) \end{aligned} \tag{1.55}$$

These operators enable to write the Weyl map (1.47) without directly using the concept of Fourier transform. They are also called Moyal quantizers [11]:

$$\hat{f} = \frac{1}{(2\pi\hbar)^n} \int dz\, f(z)\, \hat{W}(z) \tag{1.56}$$

Properties (1.55) of this system of operators clarify how the Weyl map in this form can be inverted. Its inverse, the Wigner map, is given by:

$$f(z) = Tr\left[\hat{f}\hat{W}^{\dagger}(z)\right] \tag{1.57}$$

Moreover, one has:

$$Tr\left[\hat{A}^{\dagger}\hat{B}\right] = \frac{1}{(2\pi\hbar)^n} \int dz\, A^{*}(z)\, B(z) \tag{1.58}$$

This means that Weyl-Wigner map defines a bijection between the set of square-integrable functions on the plane, and the set of Hilbert-Schmidt operators in the Hilbert space on which the Weyl system has been realized.

Although this bijection is well suited for square integrable functions, it is of interest trying to calculate what are the operators Weyl map associates to the coordinate functions. Just to keep notation clear, in this example let $M = \mathbb{R}$. For $f(q,p) = q$ one has, formally:

$$(\hat{q}\psi)(s) = \int \frac{dqdp}{\pi\hbar}\, q e^{2i(s-q)p/\hbar} \psi(2q-s) =$$

integration in dp gives a $\pi\hbar\delta(s-q)$ factor

$$= \int dq\delta(s-q)\,\psi(2q-s) \;=\; s\psi(s) \tag{1.59}$$

This shows that $\hat{Q}$ operator in the Schrödinger representation, that is the multiplication by the coordinate on the line, is the Weyl image of the coordinate function q on the phase space. For $f(q,p) = p$ one has:

$$(\hat{p}\psi)(s) \;=\; \int \frac{dqdp}{\pi\hbar} pe^{2i(s-q)p/\hbar}\psi(2q-s) =$$

the position $x = 2q - s$ brings the integral in the form:

$$= \int \frac{dxdp}{2\pi\hbar} pe^{i(s-x)p/\hbar}\psi(x) =$$

The integration over dx gives the Fourier transform of $\psi(x)$, so

$$(\hat{p}\psi)(s) = \int \frac{dp}{\sqrt{2\pi\hbar}} p\tilde{\psi}(p)\,e^{isp/\hbar}$$

This expression is clearly equal to

$$(\hat{p}\psi)(s) \;=\; -i\hbar\frac{d\psi}{ds} \tag{1.60}$$

So the coordinate function p is mapped in the $\hat{P}$ operator in the Schrödinger representation.

The next step is the study of the operator it associates to a generic monomial in the coordinate functions. It can be proved that:

$$\hat{\Omega}\left(q^a p^b\right) \;=\; \frac{1}{2^a}\sum_{k=0}^{a}\begin{pmatrix} a \\ k \end{pmatrix}\hat{Q}^k\hat{P}^b\hat{Q}^{a-k} \tag{1.61}$$

This example shows what is the ordering that the Weyl map introduces in the quantization of a sufficiently generic element of the algebra of classical observables, depending by both coordinates q and p, promoted to noncommuting variables.

1.3 The Moyal product for the noncommutative plane

The fact that the Weyl-Wigner map is invertible enables to define a different product in the space of functions on this cartesian phase space. This product is called the Moyal product:

$$\hat{\Omega}(f * g) \;=\; \hat{\Omega}(f)\,\hat{\Omega}(g) \tag{1.62}$$

It is non commutative, being a realization, in the space of functions, of the non commutative product among operators. Written in terms of functions, its integral form is:

$$(f * g)(z) = \int \frac{da}{(2\pi\hbar)^n} e^{-i\tilde{\omega}(a,z)/\hbar} \int \frac{db}{(2\pi\hbar)^n}\,\tilde{f}(b)\,\tilde{g}(a-b)\,e^{-i\tilde{\omega}(a,b)/2\hbar} \tag{1.63}$$

This product is nonlocal: this means that the support of the product $f * g$ can be non void although the intersection of the supports of the functions f and g is void. It is seen to be related to a "commutative" convolution product between symplectic Fourier transforms $\tilde{f}$ and $\tilde{g}$, but now in some sense "deformed" by the integral kernel $e^{-i\tilde{\omega}(a,b)/2\hbar}$, whose origin lies in the symplectic structure on the phase space, that plays a crucial role in the Weyl form of the commutation relations.

The Moyal product, written in the nonlocal form (1.94), can be also cast as:

$$(f * g)(z) = \frac{1}{(\pi\hbar)^{2n}} \int dtdv\, f(t+z)\, g(v+z)\, e^{2i\tilde{\omega}(t,v)/\hbar} \tag{1.64}$$

This integral form has been exploited by M.Rieffel in a remarkable monograph [35], as a starting point for a theory of general deformation of C^*- algebras[10]. The first space on which studying mathematical properties of the Moyal product is the space of Schwartzian functions $\mathcal{S}^\infty\left(\mathbb{R}^{2n}\right)$. It is possible to prove [17] that Moyal product is associative in the set of Schwartzian functions. The complex conjugation defines an involution in this space, and in the limit of $\hbar \to 0$, one has $(f * g)(z) = (fg)(z)$ for every point z in the vector space $L = \mathbb{R}^{2n}$. This can be summarised by saying that $\mathcal{A}_\hbar = \left(\mathcal{S}^\infty\left(\mathbb{R}^{2n}\right), *\right)$ is an associative, nonunital (because the function identically equal to 1 is not Schwartzian), involutive (involution is given by the complex conjugation) algebra with a continuous product.

The Moyal product can be defined, by duality, on a set larger than $\mathcal{S}^\infty$. The dual space of that of the Schwartzian functions is the space of the tempered distributions ($\mathcal{S}'$). For $F \in \mathcal{S}'\left(\mathbb{R}^{2n}\right)$, the evaluation on $f \in \mathcal{S}\left(\mathbb{R}^{2n}\right)$ can be written as a kind of scalar product $\langle F, f\rangle \in \mathbb{C}$. This notation is intended as a shorthand for the integral of the kernel of the distribution F times the function f. It is possible to define $F * f$ and $f * F$ as elements of $\mathcal{S}'$ by:

$$\langle F * f, g\rangle = \langle F, f * g\rangle \quad \langle f * F, g\rangle = \langle F, g * f\rangle \tag{1.65}$$

while the involution is extended to $\mathcal{S}'$ by:

$$\langle F^\dagger, f\rangle = \overline{\langle F, f^\dagger\rangle} \tag{1.66}$$

It is possible to consider the left and right multiplier algebras:

$$\begin{aligned} \mathcal{M}_\hbar^L &= \{F \in \mathcal{S}'\left(\mathbb{R}^{2n}\right) : F * g \in \mathcal{S}\left(\mathbb{R}^{2n}\right) \forall g \in \mathcal{S}\left(\mathbb{R}^{2n}\right)\} \\ \mathcal{M}_\hbar^R &= \{F \in \mathcal{S}'\left(\mathbb{R}^{2n}\right) : g * F \in \mathcal{S}\left(\mathbb{R}^{2n}\right) \forall g \in \mathcal{S}\left(\mathbb{R}^{2n}\right)\} \end{aligned} \tag{1.67}$$

The intersection of the two gives the multiplier algebra:

$$\mathcal{M}_\hbar = \mathcal{M}_\hbar^L \cap \mathcal{M}_\hbar^R \tag{1.68}$$

This $\mathcal{M}_\hbar$ is a complete, unital $*-$algebra. It contains the identity, the constant function, the plane waves, i.e. functions of the form e^{izw}, and even the

[10]In appendix A.1 there is an introduction to the algebraic concepts mentioned in these pages

Dirac δ function and the monomials in the coordinates. So this algebra is a compactification of $\mathcal{S}^\infty$ defined by duality.

This algebra is the *noncommutative Moyal plane.*

This analysis suggests to explicitly calculate the form of this product in the case f and g are two Schwartzian functions. It can be written as:

$$f * g = f(z)\, e^{-\frac{i\hbar}{2}\left[\frac{\overleftarrow{\partial}}{\partial q^a}\frac{\overrightarrow{\partial}}{\partial p_a}-\frac{\overleftarrow{\partial}}{\partial p_a}\frac{\overrightarrow{\partial}}{\partial q^a}\right]} g(z) \tag{1.69}$$

where the arrows over the partial derivatives symbol indicate which is the function on which they act. Equivalently:

$$f * g = f e^{-\frac{i\hbar}{2}\tilde{\Lambda}_{ab}\overleftarrow{\partial}_a \wedge \overrightarrow{\partial}_b} g \tag{1.70}$$

The fact that the Moyal product of f times g depends on all the derivatives of f and g is a different way to look at it as a nonlocal product. It is important to note that this notion of nonlocality is not equivalent to that given before. The first terms of this expansion are:

$$\begin{aligned} f * g &= f \cdot g + \frac{i\hbar}{2}\left(\frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a} - \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a}\right) + o\left(\hbar^2\right) \\ &= f \cdot g - \frac{i\hbar}{2}\{f, g\} + o\left(\hbar^2\right) \end{aligned} \tag{1.71}$$

This expression shows that Moyal product can be seen as a deformation of the usual pointwise among functions on the plane[11].

The dependence on the parameter $\hbar$ is more intuitive in this expression, as the meaning of the limit $\hbar \to 0$.

Moreover, a careful analysis shows that formula (1.69) can be extended to evaluate the product even between some functions not belonging to the space $S^\infty\left(\mathbb{R}^2\right)$. For the generators:

$$\begin{aligned} q^a * p_b &= q^a p_b - \frac{i\hbar}{2}\delta^a_b \\ p_b * q^a &= q^a p_b + \frac{i\hbar}{2}\delta^a_b \end{aligned} \tag{1.72}$$

The noncommutative plane algebra $\mathcal{M}_\hbar$ can be seen as the algebra formally generated by these noncommuting coordinates.

An algebraic type analysis can be done even from a different starting point. In the previous section it has been shown how square integrable functions on the vector space $\mathbb{R}^{2n}$ are mapped by the Weyl map onto Hilbert-Schmidt operators. So, it is natural to study the Moyal product among these functions. Properties of Hilbert-Schmidt operators show that, if f and g are in $\mathcal{L}^2\left(\mathbb{R}^{2n}\right)$, then the product $f * g$ is still square integrable: moreover, the product is continuous for $\hbar \to 0$. This algebra can be extended. One can define:

$$A_\hbar = \{F \in \mathcal{S}'\left(\mathbb{R}^{2n}\right) : F * g \in \mathcal{L}^2\left(\mathbb{R}^{2n}\right) \forall g \in \mathcal{L}^2\left(\mathbb{R}^{2n}\right)\} \tag{1.73}$$

[11] Wigner-Weyl-Moyal method, writing products and commutators of operators in phase space language, has been instrumental in giving rise to the subject of deformation quantization [6].

equipped with a norm, mutuated by the $\mathcal{L}^2$ norm:

$$\| F \|_\hbar = \sup\{\| F * g \|_{\mathcal{L}^2} / \| g \|_{\mathcal{L}^2} : 0 \neq g \in \mathcal{L}^2\left(\mathbb{R}^{2n}\right)\} \tag{1.74}$$

This algebra $(A_\hbar, \| \cdot \|_\hbar)$ is proved to be a unital C^*-algebra, isomorphic to the set of bounded operators $\mathcal{B}\left(\mathcal{L}^2\left(\mathbb{R}^n\right)\right)$. It is interesting to note that the Weyl-Wigner isomorphism defines a realization of the GNS construction for this algebra [44]. The Algebra $A_\hbar$ contains integrable functions $\mathcal{L}^1\left(\mathbb{R}^{2n}\right)$ and plane waves, but it does not contain non constant polynomials.

Since plane waves belong both to $A_\hbar$ and $\mathcal{M}_\hbar$, the Moyal product for two plane waves of covectors u and w is given by

$$e^{iuz} * e^{iwz} = e^{i\tilde{\Lambda}(u,w)/2\hbar} e^{i(u+w)z} \tag{1.75}$$

So plane waves close an algebra, the Weyl algebra, that represents an action of the translation group on the vector space $\mathbb{R}^{2n}$. This action is put in a more intuitive form if a "symplectic" plane wave basis is used:

$$e^{i\tilde{\omega}(u,z)/\hbar} * e^{i\tilde{\omega}(w,z)/\hbar} = e^{i\tilde{\omega}(u,w)/2\hbar} e^{i\tilde{\omega}(u+w,z)/\hbar} \tag{1.76}$$

1.4 The classical limit of quantum mechanics in the Weyl-Wigner formalism

In the previous section it has been analysed the nature of the Moyal product, and of the noncommutative algebra structure it gives to the set of functions on the cartesian phase space.

The skewsymmetrised form of this product gives:

$$\begin{aligned} \{f,g\}_M &\equiv \frac{i}{\hbar}(f * g - g * f) \\ &= \{f,g\} + o(\hbar) \end{aligned} \tag{1.77}$$

and this is called Moyal bracket. It is the bilinear map that translates, in the set of functions, the notion of commutator in the set of operators. This is the reason why this map is bilinear, satisfies the Jacobi identity, and satisfies a Leibniz rule with respect to the Moyal product:

$$\{f * g, h\}_M = f * \{g,h\}_M + \{f,h\}_M * g \tag{1.78}$$

Relations (1.77) define a deformation of the Poisson structure in the set of functions on the phase space.

In the space of functions on the plane, the introduction of Moyal bracket enables to write:

$$\begin{aligned} \{q^a, p_b\}_M &= \delta^a_b \\ \{q^a, H\}_M &= \frac{\partial H}{\partial p_a} \\ \{p_a, H\}_M &= -\frac{\partial H}{\partial q^a} \end{aligned} \tag{1.79}$$

Whatever the function H were, the Moyal bracket with coordinate functions gives the same result the Poisson bracket would give. Derivations associated with the coordinate functions are the same both in the classical algebra and in the "deformed" algebra. So, what is the meaning of the expression for a generic f?

In classical formalism a theorem already mentioned says that smooth derivations for the abelian algebra of functions on the phase space are represented by vector fields, and vector fields are infinitesimal generators of classical dynamics. So "classical" derivations are related to the classical evolution of the observables.

If the same set of functions is given a non abelian algebraic structure via the Moyal product, then the Moyal bracket provides a class of derivations of this algebra. A function on the phase space represents, via the Moyal bracket, a derivation of the quantum algebra. These derivations are related to quantum dynamics.

If one considers the quantum evolution in the Heisenberg picture, then operators are evolved:

$$\hat{A}(t) = \hat{U}^{\dagger}(t)\,\hat{A}\hat{U}(t) \tag{1.80}$$

and, applying the Wigner map to both sides, it can be written in terms of an evolution of the symbols:

$$A(t) = U^{*}(t) * A * U(t) \tag{1.81}$$

If one considers the infinitesimal form of this relation, with a quantum evolution operator in the form $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$, this equation is written:

$$\begin{aligned} \frac{d}{dt}A(t) &= \{H, A(t)\}_{M} \\ &= \{H, A(t)\} + o(\hbar) \end{aligned} \tag{1.82}$$

Derivations given by an element of the algebra via the Moyal bracket represent the infinitesimal form of a quantum dynamics in the Heisenberg picture.

Moreover, since the Moyal bracket is a deformation of the Poisson bracket, this equation shows that the Weyl-Wigner formalism enables to write the quantum evolution in terms of equations involving functions on the phase space, carrying a classical dynamics, in such a way to recover, in the limit $\hbar \to 0$, the classical evolution for the Weyl symbols in the Poisson formalism.

This is the meaning of the classical limit procedure in the Weyl-Wigner formalism.

1.5 Generalizing Weyl systems

In the previous sections, a Weyl system has been defined (1.20) in terms of a generic, though translationally invariant, symplectic form on the vector phase space L. Nevertheless the explicit realization of the Schrödinger representation and of the Weyl-Wigner maps have been studied in the case of the symplectic form ω being in the canonical Darboux form $\tilde{\omega}$. The aim of this section is to generalize this explicit realization of a Weyl system. The first generalization will be introduced to cover the case of a symplectic structure ω which is no more in the canonical form. The second generalization will be the study of a Weyl system, defined as a unitary projective representation of the abelian group of translations, with phase factors no longer simply corresponding to a symplectic form.

1.5.1 Weyl systems for translationally invariant symplectic structures

To consider this first generalization of the Weyl system's notion, it is useful to review the topics covered in the first two subsections (1.1.3) and (1.1.2). Let $\hat{D}(z)$ be a Weyl system (1.31) for $(L, \tilde{\omega})$. Let T be an automorphism (an invertible and linear map) in the vector space L. It does not need to be symplectic: $T \in \mathcal{A}ut(L)$. Acting with such a transformation on L, one has:

$$\hat{D}(Tz + Tu) = \hat{D}(Tz)\,\hat{D}(Tu)\,e^{\frac{i}{2\hbar}\tilde{\omega}(Tz,Tu)} \tag{1.83}$$

by linearity:

$$\hat{D}(T(z+u)) = \hat{D}(Tz)\,\hat{D}(Tu)\,e^{\frac{i}{2\hbar}\omega(z,u)} \tag{1.84}$$

In this relations it has been considered that the transformation of vectors of L by such a T, can be dually read as a transformation of the symplectic structure:

$$\omega(z,u) \equiv \tilde{\omega}(Tz, Tu) \;\rightarrow\; \omega = T^t\,\tilde{\omega}\,T \tag{1.85}$$

Now it is possible to define:

$$\hat{D}(Tz) \equiv \hat{D}_T(z) \tag{1.86}$$

obtaining:

$$\hat{D}_T(z+u) = \hat{D}_T(z)\,\hat{D}_T(u)\,e^{\frac{i}{2\hbar}\omega(z,u)} \tag{1.87}$$

In this approach, $\hat{D}_T(z)$ is a Weyl system for (L, ω) with ω obtained as (1.85). Of course, $\hat{D}_T(z)$ is recovered as a standard Weyl system if the automorphism T is also symplectic. Moreover, the general analysis developed in the previous sections clarifies that, if $T' = T_S T$ (T' is the composition of the automorphism T with the symplectic automorphism T_S) then the Weyl system $\hat{D}_{T'}(z)$ is unitarily equivalent to $\hat{D}_T(z)$.

Properties of invertible skewsymmetric matrices whose coefficient are constant, which represents translationally invariant symplectic forms on the vector space L, show that the problem of realizing a Weyl system for (L, ω) is solved by $\hat{D}_T(z) = \hat{D}(Tz)$ where $\hat{D}(z)$ is a Weyl system for $(L, \tilde{\omega})$ (1.31), and T is the automorphism that solves the equation $T^t\tilde{\omega}T = \omega$. Such a T is defined by this requirement and recovered up to the composition with an arbitrary symplectic automorphism.

The problem of quantizing the dynamics of a particle in a constant background magnetic field can be studied with these tools. Classical evolution is formalized on $T^*\mathbb{R}^3$ in terms of a symplectic structure with constant coefficients depending by the components of the $\vec{B}$ fields, and a quadratic hamiltonian function. Equations of motions are:

$$\begin{aligned} \dot{q}^i &= p^i \\ \dot{p}_i &= \epsilon_i^{jk} p_j B_k \end{aligned} \tag{1.88}$$

for position coordinates q^i and gauge invariant momenta coordinates p_i. These equations define an Hamiltonian vector field with respect to the Poisson structure defined by:

$$\begin{aligned}
\{q^i, q^j\} &= 0 \\
\{q^i, p_j\} &= \delta^i_j \\
\{p_i, p_j\} &= \epsilon^k_{ij} B_k
\end{aligned} \tag{1.89}$$

whose symplectic counterpart is:

$$\omega = -\epsilon_{ijk} B^k dq^i \wedge dq^j + dq^i \wedge dp_i \tag{1.90}$$

The realization of a Weyl system for this symplectic structure gives rise to a quantum Hamiltonian operator defined without any reference to the vector potential $\nabla \times \vec{A} = \vec{B}$, as it is the case in the standard approach to quantization in terms of the so called minimal coupling procedure. Even an ambiguity in the definition of this Hamiltonian operator, due to the gauge transformation properties of the vector potential, can be recovered inside this formalism: it is related to the invariance of solution for a Weyl system by symplectic transformations.

This generalization of a Weyl system can be naturally brought into a generalization of the Weyl map, and the Moyal product. Formula (1.47) can be extended to the case just shown:

$$\hat{\Omega}(f) = \int \frac{dw}{(2\pi\hbar)^n} |T| \tilde{f}(w) \hat{D}_T(w) \tag{1.91}$$

f is a function on (L, ω) The symplectic Fourier transform (appendix A.2)$\tilde{f}$ is obtained via an automorphism T that brings the symplectic form ω in the Darboux form $\tilde{\omega}$: $T^t \tilde{\omega} T$. This transformation T is the same that can be used to define the Weyl system for (L, ω): $|T|$ is the determinant of the matrix representing T.

This formula for the Weyl map simply shows that the role of the T transformation is to cast the problem in the form of defining the Weyl map for functions on a vector space in a coordinate chart in which the symplectic structure has the Darboux form.

Even this Weyl map is invertible, and the Wigner map is given by:

$$\tilde{f}(w) = Tr\left[\hat{f}\hat{D}^\dagger(w)\right] \tag{1.92}$$

This bijection now can be used to define a Moyal product in a space of functions on the space S. The relation

$$\hat{\Omega}(f * g) = \hat{\Omega}(f)\hat{\Omega}(g) \tag{1.93}$$

acquires the nonlocal form:

$$\begin{aligned}
(f * g)(z) &= \frac{|T|^4}{(2\pi\hbar)^{4n}} \int dx dy \int d\beta d\alpha\, f(x)\, g(y)\; e^{-i\omega(x,\beta)/\hbar} e^{-i\omega(y,\alpha-\beta)/\hbar} e^{-i\omega(\alpha,z)/\hbar} e^{i\omega(\beta,\alpha)/2\hbar} \\
&= \frac{|T|^2}{(2\pi\hbar)^{2n}} \int d\alpha\, e^{-i\omega(\alpha,z)/\hbar} \int d\beta\, \tilde{f}(\beta)\, \tilde{g}(\alpha-\beta)\; e^{i\omega(\beta,\alpha)/2\hbar}
\end{aligned} \tag{1.94}$$

This product generalizes the product (1.63) for a generic translationally invariant symplectic 2-form. This generalization can be seen also in the asymptotic expansion. On a suitable domain of functions, this product can be written in a differential form, that is the generalization of (1.69)

$$f * g \;=\; f\, e^{-\frac{i}{2\hbar}\left(\overleftarrow{\partial}_a \Lambda_{ab} \overrightarrow{\partial}_b\right)}\, g \tag{1.95}$$

Here the Poisson tensor is related to the inverse of the 2-form represented by ω. One of the reason why it has been used the definition of symplectic Fourier transform is that, if it were not, all this procedure would have not been covariant for the whole symplectic group. Moreover, the asymptotic expansion of the product (1.95) would not have been an exponentiation of the Poisson bivector, thus eliminating the possibility to generalize the analysis culminated in (1.82) on the classical limit.

1.5.2 Weighted Weyl systems

In the previous sections, the space L has been looked at as a real, even dimensional linear space. To proceed along the path of studying a generalization of the concept of Weyl systems, L can be now considered as a realization of the abelian group of translations $\left(\mathbb{R}^{2n}, +\right)$, while $\hat{D}$ as a projective unitary representation of this group, where the phase factors are given by the symplectic structure.

It is natural to consider now the definition of a more general unitary representation for this group:

$$\hat{D}_\Phi\left(z+u\right) \;=\; e^{i\Phi(z,u)/2\hbar}\, \hat{D}_\Phi\left(z\right)\, \hat{D}_\Phi\left(u\right) \tag{1.96}$$

The obvious demand that this representation preserves the associativity of group composition forces tha phase factors to satisfy a peculiar condition:

$$\Phi\left(z, u+v\right) + \Phi\left(u, v\right) \;=\; \Phi\left(z, u\right) + \Phi\left(z+u, v\right) \tag{1.97}$$

Without entering into a cohomological characterization of this relation, it is enough to say that such a Φ is called a cocycle. It is important to note that, if $\Phi\left(z, u\right)$ is linear in both entries, then it is necessarily a cocycle. Following this analysis, it is clear that a standard Weyl system (1.20) is such a representation, in which the group is even dimensional, and the cocycle is a skewsymmetric nondegenerate bilinear function.

A generalization of that construction is given by a choice of $\Phi\left(z, u\right)$ with a nondegenerate skewsymmetric part, and a nonvanishing symmetric part:

$$\Phi\left(z, u\right) \;=\; A\left(z, u\right) + S\left(z, u\right)$$

where:

$$A\left(z, u\right) \;=\; \frac{1}{2}\left[\Phi\left(z, u\right) - \Phi\left(u, z\right)\right]$$
$$S\left(z, u\right) \;=\; \frac{1}{2}\left[\Phi\left(z, u\right) + \Phi\left(u, z\right)\right]$$

Now it is possible to follow the same path developed in the study of standard Weyl systems. These definitions enable to write:

$$\hat{D}_\Phi\left(\alpha z\right) \hat{D}_\Phi\left(\beta z\right) \;=\; \hat{D}_\Phi\left(\beta z\right) \hat{D}_\Phi\left(\alpha z\right) \tag{1.98}$$

$$\hat{D}_\Phi\left((\alpha+\beta)\, z\right) \;=\; e^{i\alpha\beta S(z,z)/2\hbar}\, \hat{D}_\Phi\left(\alpha z\right) \hat{D}_\Phi\left(\beta z\right) \tag{1.99}$$

This means that, restricted on a one dimensional subspace, $\hat{D}_\Phi$ is no longer a faithful representation of the additive line. $\hat{D}_\Phi\left(\alpha z\right)$ is not anymore a one parameter group of unitary operators. Stone's theorem cannot be invoked to define generators to identify with physical observables.

Using as a guide the standard Weyl systems theory, it is natural to define a set of hermitian operators $\hat{G}\left(z\right)$ depending on an element of the group, and a real function of two variables $w\left(\alpha, z\right)$ by the relation:

$$\hat{D}_\Phi\left(\alpha z\right) \;=\; e^{i\left[\alpha \hat{G}(z) + w(\alpha, z)\right]/\hbar} \tag{1.100}$$

Equation (1.99) is satisfied if:

$$w\left(\alpha+\beta, z\right) - w\left(\alpha, z\right) - w\left(\beta, z\right) \;=\; \frac{1}{2}\, S\left(z, z\right)\, \alpha\beta \tag{1.101}$$

Moreover, it can be seen that this function w should satisfy a sort of homogeneity condition in the z variable:

$$w\left(\alpha+\beta, \gamma z\right) - w\left(\alpha, \gamma z\right) - w\left(\beta, \gamma z\right) \;=\; \frac{\gamma^2}{2}\, S\left(z, z\right)\, \alpha\beta \tag{1.102}$$

If one tries to obtain the commutation relations among so defined hermitian "generators", from the definition of Weyl systems:

$$\begin{aligned} \hat{D}_\Phi\left(\alpha z + \beta u\right) \;&=\; e^{i\alpha\beta\Phi(z,u)/2\hbar}\, \hat{D}_\Phi\left(\alpha z\right) \hat{D}_\Phi\left(\beta u\right) \\ \hat{D}_\Phi\left(\alpha z + \beta u\right) \;&=\; e^{i\alpha\beta\Phi(u,z)/2\hbar}\, \hat{D}_\Phi\left(\beta u\right) \hat{D}_\Phi\left(\alpha z\right) \end{aligned}$$

one obtains:

$$\left[\hat{G}\left(z\right), \hat{G}\left(u\right)\right] \;=\; i\hbar A\left(z, u\right) \tag{1.103}$$

This indicates that commutation rules among generators depend only by the skewsymmetric part of the cocycle factor. The solution of equation (1.101) is:

$$w\left(\alpha, z\right) \;=\; \frac{\alpha^2}{4}\, S\left(z, z\right) \tag{1.104}$$

Campbell-Baker-Hausdorff formula enables to cast a Weyl system in the form:

$$\hat{D}_\Phi\left(z\right) \;=\; e^{i\left[z^a \hat{G}(e_a) + w(z^a, e_a)\right]/\hbar} \tag{1.105}$$

The generalization of the notion of Weyl system to the case of a generic bilinear cocycle for the translation group is then of the form:

$$\hat{D}_\Phi\left(z\right) \;=\; \hat{D}\left(z\right)\, e^{iS(z,z)/4\hbar} \tag{1.106}$$

Here $\hat{D}\left(z\right)$ is a Weyl system for a symplectic structure given by the skewsymmetric part of the cocycle Φ. The extra term can be seen as a weight, depending only on the symmetric part of the cocycle.

1.5.3 Weighted Weyl map

The generalization studied in the previous subsection is very important, because it enables to define a different Weyl map, which means a different ordering in going from commutative variables to noncommutative ones, for the algebra of functions on the space $\mathbb{R}^{2n}$. In the following it will be considered the case of a cocycle Φ whose skewsymmetric part is in the canonical form.

A Weyl map is now generalised to be:

$$\begin{aligned}\hat{\Omega}_{\Phi}(f) &= \int \frac{du}{(2\pi\hbar)^n}\,\tilde{f}(u)\,\hat{D}_{\Phi}(u)\\ &= \int \frac{du}{(2\pi\hbar)^n}\,\tilde{f}(u)\,e^{iS(u,u)/4\hbar}\,\hat{D}(u)\end{aligned} \tag{1.107}$$

Also this map can be inverted:

$$\tilde{f}(u) = e^{-iS(u,u)/4\hbar}\,Tr\left[\hat{\Omega}_{\Phi}(f)\,\hat{D}^{\dagger}_{\Phi}(u)\right] \tag{1.108}$$

This relation is based on the fact that operators $\hat{D}_{\Phi}$, being the product of standard $\hat{D}$ times a phase, close a relation of the same kind of (1.51). Two examples are interesting: just to symplify notations, the vector space considered will be of dimension 2, and vector z will have components (q,p)

The first is the case when the symmetric matrix S is given by:

$$S = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

It can be seen that the coordinate function q is mapped into the generator $\hat{Q}$ of a standard Weyl system, and the coordinate function p is mapped into the standard generator $\hat{P}$. But this new quantizing map defines a peculiar ordering for images of monomials (cfr(1.61)):

$$\hat{\Omega}_{\Phi}\left(q^a p^b\right) = \hat{P}^b\hat{Q}^a \tag{1.109}$$

The noncommutative product one obtains in the space of functions on the plane, via this *weighted* Weyl map, is, in the formal expansion valid on a suitable domain:

$$\begin{aligned}(f *_S g)(q,p) &= \sum_{k=0}^{\infty} \frac{(-i\hbar)^k}{k!}\,\frac{\partial^k f}{\partial q^k}\,\frac{\partial^k g}{\partial p^k}\\ (f *_S g)(q,p) &= f\,e^{-i\hbar\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p\right)}\,g\end{aligned} \tag{1.110}$$

The second is the case when the symmetric matrix S' is given by:

$$S' = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

Even in this case, it can be seen that the coordinate function q is mapped into the generator $\hat{Q}$ of a standard Weyl system, and the coordinate function p is mapped into the standard generator $\hat{P}$. Now the ordering this quantizing map defines is, on images of monomials (1.61):

$$\hat{\Omega}_{\Phi}\left(q^a p^b\right) = \hat{Q}^a\hat{P}^b \tag{1.111}$$

The noncommutative product one obtains in the space of functions on the plane, via this *weighted* Weyl map, is, in the formal expansion valid on a suitable domain:

$$
\begin{aligned}
(f *_{S'} g)(q,p) &= \sum_{k=0}^{\infty} \frac{(i\hbar)^k}{k!} \frac{\partial^k f}{\partial p^k} \frac{\partial^k g}{\partial q^k} \\
(f *_{S'} g)(q,p) &= f\, e^{-i\hbar\left(\overleftarrow{\partial}_p \overrightarrow{\partial}_q\right)} g
\end{aligned}
\tag{1.112}
$$

It is very interesting to note that this two deformed products are equivalent to the standard Moyal product,when the equivalence is defined by the realization of an operator such that:

$$
\begin{aligned}
T^{(S)} &: \left(\mathcal{F}\left(\mathbb{R}^2\right), *_S\right) \mapsto \left(\mathcal{F}\left(\mathbb{R}^2\right), *\right) \\
T^{(S)}(f *_S g) &= \left(T^{(S)} f\right) * \left(T^{(S)} g\right)
\end{aligned}
\tag{1.113}
$$

The operators, for these two cases, can be proved to be [53]:

$$
\begin{aligned}
T^{(S)} &= \sum_{n=0}^{\infty} \left(\frac{i\hbar}{2}\right)^n \frac{1}{n!} \left(\frac{\partial}{\partial p}\right)^n \left(\frac{\partial}{\partial q}\right)^n \\
T^{(S')} &= \sum_{n=0}^{\infty} \left(-\frac{i\hbar}{2}\right)^n \frac{1}{n!} \left(\frac{\partial}{\partial p}\right)^n \left(\frac{\partial}{\partial q}\right)^n
\end{aligned}
\tag{1.114}
$$

1.6 Weyl map from coherent states for the Heisenberg-Weyl-Wigner group

Throughout this chapter, it has been pointed out how the Weyl-Wigner formalism can be studied stressing the accent on a group theoretical approach. A standard Weyl system has been realized by Displacement operators, as a unitary projective representation of the translation group, where phase factors are related to the symplectic structure of the linear space on which it acts. This clarifies some aspects of the deep contact between geometrical foundations in the formulation of classical and quantum dynamics. Even the way the first generalization of section 1.5.2 has been presented, goes towards an analysis of a more general class of representations for the same group.

In section 1.2 Weyl map has been written also using a set of Weyl operators $\hat{W}(z)$, whose properties are summarized in (1.55). Displacement operators have been defined via their composition properties. What is the composition rule for this system of Weyl operators? It can be checked:

$$
\begin{aligned}
\hat{W}(z)\, \hat{W}(z') &= 4^n\, e^{2i\tilde{\omega}(z,z')/\hbar} \hat{D}(z-z') \\
&= 2^n\, e^{2i\tilde{\omega}(z,z')/\hbar} \hat{W}(z-z')\, \hat{\mathcal{P}}
\end{aligned}
\tag{1.115}
$$

This relation says that Weyl operators do not define a group. But the introduction of the system of $\hat{W}(z)$ operators acquires an interesting geometrical meaning if it is seen in the perspective of a representation of the so called Heisenberg-Weyl-Wigner (HWW) group.

In the previous sections of this chapter, to stress that the formalism was born to study the problem of quantization for a classical dynamics, and classical limit for a quantum dynamics, the role of $\hbar$ has been kept explicitly. From an algebraic point of view, which is the one noncommutative geometry starts from, $\hbar$ is just a parameter. It is the parameter that represents a non commutativity in the quantum relations for canonical observables, and it has been considered as a deformation parameter in a formalism developed to unify both the classical and the quantum ones.

In this section, since the stress will be just on geometrical aspects of the formalism, an identification of the quantities with physical observables will be abandoned. The deformation parameter will be a constant θ, and the space L will be the space $\mathbb{R}^2$ with coordinates q and p without a dimension of position or momentum. And this is at the light of future themes of this dissertation, where this Weyl formalism will be used to study some specific non commutative spaces.

Canonical commutation relations define the Lie algebra of the Heisenberg-Weyl (HW) group. The group manifold is $\mathbb{R}^3$, and elements of the group are labelled as triples (q, p, λ). The composition rule is:

$$(q, p, \lambda) \cdot (q', p', \lambda') = \left(q + q', p + p', \lambda + \lambda' + \frac{1}{2}(qp' - q'p)\right) \tag{1.116}$$

Now the idea is to define a new group, obtained as a semi-direct product of HW with the group $\mathbb{Z}_2$.

Among triples of the form (ξ, λ, α) (where ξ is a complex number, and represents a point in a complex plane, λ is a real number, and α can take the discrete values ± 1) it is possible to define a composition by:

$$(\xi, \lambda, \alpha) \cdot (\xi', \lambda', \alpha') \equiv \left(\xi + \alpha\xi', \lambda + \lambda' + \frac{i}{2\theta}\alpha\left(\bar{\xi}\xi' - \bar{\xi}'\xi\right), \alpha\alpha'\right) \tag{1.117}$$

Then the set acquires the structure of a group, that is topologically equivalent to 2 copies of $\mathbb{R}^3$. This is called Heisenberg-Weyl-Wigner (HWW) group. The identity element is

$$\mathbf{1}_{W'} = (0, 0, 1) \tag{1.118}$$

and the inverse of a generic element is:

$$(\xi, \lambda, \alpha)^{-1} = (-\alpha\,\xi, -\lambda, \alpha) \tag{1.119}$$

The next step is to define a system of coherent states for this group. To this extent, a unitary irreducible representation of it on a Hilbert space should be considered. Following Bargmann and Fock [31], one can introduce a space of functions which are complex analytical in a w variable, endowed with a scalar product:

$$\langle f \mid g\rangle \equiv \int \frac{d^2 w}{\pi\theta}\, e^{-\bar{w}w/\theta}\, \bar{f}(w)\, g(w) \tag{1.120}$$

The Fock space $\mathcal{F}$ will be defined as the set of those functions whose norm, resulting from this scalar product, is finite. This can be proved to be a Hilbert space. On this space a set of operators is defined: $\hat{W}'(\xi, \lambda, \alpha)$ whose action on $f \in \mathcal{F}$ is given by:

$$\left(\hat{W}' f\right)(w) = e^{i\lambda} e^{-\bar{\xi}\xi/2\theta} e^{\xi w/\theta} f\left(\alpha\left(w - \bar{\xi}\right)\right) \tag{1.121}$$

In these definitions a parameter θ has been introduced. It can be thought to have the dimension of the square of w and ξ variables, while λ is considered adimensional. In this space the most natural orthonormal basis is

$$\psi_n(w) = \frac{w^n}{\sqrt{\theta^n n!}} \tag{1.122}$$

To define a system of coherent states, a *fiducial vector* in $\mathcal{F}$ must be chosen. The easiest choice is, of course, $\psi_0(w)$. The action of $\hat{W}'$ operators on this vector gives a new set of vectors in $\mathcal{F}$. Among elements of the group, there are some whose action via the representation gives just the ψ_0 multiplied by a phase: these elements constitute the so-called isotropy subgroup of the HWW group for the chosen fiducial state. But vectors which differ by a phase can be identified,as physical states, from a quantum mechanical point of view. The quotient of these set of states by this relation gives a set of equivalence classes, the coherent states. What can be proved is that each equivalence class can be labelled by a complex number, so that the quotient space can be seen as a complex plane. This means that there is a coherent state for each point on a plane, whose explicit form is

$$\mid\xi\rangle \;\rightarrow\; \psi_\xi(w) = e^{-\bar{\xi}\xi/2\theta}e^{\xi w/\theta} \tag{1.123}$$

The coherent state labelled by the point ξ in the complex plane is an element of $\mathcal{F}$, so it is represented as a analytical function of w whose form is exactly $\psi_\xi(w)$. It is possible to prove that this system of coherent states is overcomplete:

$$\mathbf{1} = \int \frac{d^2\xi}{\pi\theta} \mid\xi\rangle\langle\xi\mid \tag{1.124}$$

and also, with ψ_n an element of the basis already considered:

$$\langle\xi\mid\psi_n\rangle = e^{-\bar{\xi}\xi/2\theta}\frac{\bar{\xi}^n}{\sqrt{n!\theta^n}} \tag{1.125}$$

$$\langle\xi\mid f\rangle = f\left(\bar{\xi}\right)e^{-\bar{\xi}\xi/2\theta} \tag{1.126}$$

where f is an element in $\mathcal{F}$.

Now an action of the HWW group on the complex plane can be defined. It is given by:

$$(\xi,\lambda,\alpha)\cdot w = \xi + \alpha w \tag{1.127}$$

It can be seen that the element $(0,\lambda,-1)$ defines a reflection of the point w with respect to the origin of the plane, while the element $(2\xi',\lambda,-1)$ defines a reflection with respect to the point ξ' of the complex plane. The image of these reflection operators, in the Fock representation, is the function:

$$\hat{W}'(2\xi',\lambda,-1)\mid\xi\rangle \;\rightarrow\; e^{i\lambda}e^{-2\bar{\xi}'\xi/\theta}e^{-2\xi' w/\theta}e^{-\bar{\xi}\xi/2\theta}e^{\xi\left(2\bar{\xi}'-w\right)/\theta} \tag{1.128}$$

In the Fock space $\mathcal{F}$ the ladder operators:

$$\hat{Z}^\dagger\mid\psi_n\rangle = \sqrt{n}\mid\psi_{n-1}\rangle \tag{1.129}$$

$$\hat{Z}\mid\psi_n\rangle = \sqrt{n+1}\mid\psi_{n+1}\rangle \tag{1.130}$$

are just creation-annihilation operators. In the representation of elements in $\mathcal{F}$ as analytic functions, they have the form:

$$\begin{aligned} \left(\hat{Z}^{\dagger} f\right)(w) &= \sqrt{\theta}\frac{df}{dw} \\ \left(\hat{Z} f\right)(w) &= \frac{1}{\sqrt{\theta}} w f(w) \end{aligned} \tag{1.131}$$

Now it is possible to write everything in the standard Hilbert space of the Schröedinger-Von Neumann representation, that of the square integrable functions on the line with respect to the translationally invariant Lebesgue measure $\mathcal{H} = \mathcal{L}^2\left(\mathbb{R}, dx\right)$

The equation (1.121) clarifies that the operators defining the unitary representation can be written as:

$$\hat{W}'(\xi, \lambda, \alpha) = e^{i\lambda} e^{-\bar{\xi}\xi/2\theta} e^{\xi\hat{Z}/\sqrt{\theta}} e^{-\bar{\xi}\hat{Z}^{\dagger}/\sqrt{\theta}} \hat{\Pi}_{\alpha} \tag{1.132}$$

In this expression,$\hat{\Pi}_{\alpha}$ is the identity operator if $\alpha = 1$, or the parity operator if $\alpha = -1$.

On $\mathcal{H}$, one can consider the operator $\hat{W}'(\xi, 0, \alpha)$ realized in terms of standard creation-annihilation operators:

$$\hat{W}'(\xi, 0, \alpha) = e^{\left(\xi\hat{a}^{\dagger} - \bar{\xi}\hat{a}\right)/\sqrt{\theta}} \hat{\Pi}_{\alpha} \tag{1.133}$$

Now, identifying:

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2\theta}}\left(\hat{Q} + i\hat{P}\right) \\ \hat{a}^{\dagger} &= \frac{1}{\sqrt{2\theta}}\left(\hat{Q} - i\hat{P}\right) \end{aligned} \tag{1.134}$$

$$\begin{aligned} \xi &= \frac{1}{\sqrt{2}}(q + ip) \\ \bar{\xi} &= \frac{1}{\sqrt{2}}(q - ip) \end{aligned} \tag{1.135}$$

we have an explicit realization of a Weyl system, in the form of a Displacement operator, composed with a parity operator:

$$\hat{W}'(\xi, 0, \alpha) = e^{\frac{i}{\theta}\left(p\hat{Q} - q\hat{P}\right)} \hat{\Pi}_{\alpha} \tag{1.136}$$

This means that a Weyl operator is related to the representation of the elements of the Heisenberg-Weyl-Wigner group which, acting on a plane, define a reflection [27]:

$$\hat{W}(q, p) = 2\hat{W}'(2\xi, 0, -1) \tag{1.137}$$

Chapter 2

Weyl-Wigner formalism for compact Lie groups

In the first chapter an introduction to the Weyl-Wigner formalism has been presented. It has been analysed the case where the classical phase space is a cartesian vector space equipped with a translationally invariant symplectic structure. In particular, the phase space has been seen as the manifold representing the abelian group of translations, and the noncommutativity, parametrized by $\hbar$, has been introduced by an explicit use of the symplectic structure. In section 1.5.2 the noncommutativity of the quantum observables has been formalized by the skewsymmetric term of the cocycle factor of the representation of this abelian group. The aim of this chapter is to study a generalization of the Weyl-Wigner formalism to the case where the classical configuration space is no more a vector space, thus identifiable with the noncompact abelian group of translations, but a generic compact simple Lie group.

The chapter begins with a description of Wigner distributions in the cartesian case: they are introduced using the machinery previously developed. This section could also be considered as the end of the first chapter. It is here because the notion of Wigner distribution will be used as a guide in constructing the Weyl-Wigner isomorphism in this case.

This isomorphism should take place between a set of operators on a Hilbert space, and a set of functions on the classical phase space, which is the cotangent bundle T^*G of a compact simple Lie group G. The path followed in the first chapter would suggest, to consider, first of all, a notion of Fourier transform for functions on this space. The second step would be the definition of a kind of Weyl system for the dual of the classical phase space, and then to define a Weyl map, and a Wigner map, by the well known procedure. Section 2.2 shows what are the results of following this path in the easiest case of $G = U(1) \approx S^1$. It shows that a Weyl map for functions on a cylinder (which is the manifold T^*S^1) cannot be obtained in this way.

Nevertheless it shows that a kind of Wigner map can be defined, introducing a set of operators that generalizes the properties of Moyal quantizers (1.55). These operators are used to define a map from the space of operators to the space of functions (symbols) defined on $S^1 \times \mathbb{Z}$. The symbols of density matrix operators are functions whose marginals distributions reproduce the expected probability distribution for the quantum mechanical system usually referred to as a particle constrained to move on a circle. These symbols are then called Wigner distributions, the map is called Wigner map, and Weyl map is obtained as its inverse. The novelty of this approach [29] is that this Weyl-Wigner isomorphism is defined between the set of operators on a Hilbert space, and the set of functions on the space $S^1 \times \mathbb{Z}$. This space can be seen as a *quantum cotangent bundle* of the circle S^1.

In section 2.3 this approach is developed in detail for the case where the configuration space of a classical system is a compact simple Lie group. First of all the classical kinematics is analysed, to set up its quantum version, which generalizes the canonical commutation relations introduced by Dirac, and already studied in detail. The noncommutativity of quantum observables can be traced back to the non abelianess of the group G, not related to a noncommutativity constant parametrized by $\hbar$. It will not have any role in the following studies. Harmonic analysis on the group G suggests what is the space on which Wigner distributions are defined, and what are the quantizer operators. Then the complete Weyl-Wigner isomorphism is deduced. On the space of symbols (that are now functions on the quantum cotangent space $G \times \Gamma$) a noncommutative product can be set.

The last section shows how this formalism can be used to define, when the group G is nonabelian, a Weyl-Wigner isomorphism between operators and functions on the *classical* cotangent space, thus solving the initial problem of generalising the Weyl-Wigner isomorphism to a set of classical systems larger than the cartesian ones.

2.1 From Weyl map to Wigner functions

In the standard quantum formalism an observable is formalized via a self-adjoint operator on a separable Hilbert space, whose rays represent the physical pure states. The set of measured values for an observable, represented for example by the operator $\hat{A}$, if the system is in a state represented by a normalized ket $|\psi\rangle$, is an experimental distribution whose mean value is formalized as (in this analysis, it will be considered $L = \mathbb{R}^2 = T^*\mathbb{R}$):

$$\langle \hat{A} \rangle = \langle \psi \mid \hat{A} \mid \psi \rangle \tag{2.1}$$

The right hand side of this equation can be written, on a suitable set of operators, as:

$$\langle \hat{A} \rangle = Tr\left[\hat{A} \mid \psi\rangle\langle\psi \mid\right] \tag{2.2}$$

In the Weyl formalism, the mapping between functions and operators is such that relation (1.58) is valid, so that it is possible to write the mean value of an observable as:

$$\langle \hat{A} \rangle = Tr\left[\hat{A} \mid \psi\rangle\langle\psi \mid\right] = \int \frac{dqdp}{2\pi\hbar} A(q,p)\, W_\psi(q,p) \tag{2.3}$$

where $W_\psi(q,p)$ is called Wigner distribution [49] function for the pure state $\mid \psi\rangle$. It is the Weyl symbol of the projector $\mid \psi\rangle\langle\psi \mid$, while $A(q,p)$ is the Weyl symbol for $\hat{A}$. In general, for a density operator $\hat{\rho}$:

$$W_{\hat{\rho}}(q,p) = \int \frac{dxdk}{2\pi\hbar} e^{-i(xp-kq)/\hbar}\, Tr\left[\hat{\rho}\hat{D}^\dagger(x,k)\right] \tag{2.4}$$

It can be written also using the system of Weyl operators, defined in (1.55), by equation (1.57):

$$W_{\hat{\rho}}(q,p) = Tr\left[\hat{\rho}\hat{W}^{\dagger}(q,p)\right] \tag{2.5}$$

In the Schrödinger realization, where $\psi(s)$ is the wave function representing the ket state $\mid\psi\rangle$, they are given by:

$$W_{\psi}(q,p) = \int ds\, e^{-ips/\hbar}\,\psi^{*}(q-s/2)\,\psi(q+s/2) \tag{2.6}$$

while for the density operator $\hat{\rho} = \mid\phi\rangle\langle\psi\mid$: (both $\mid\phi\rangle$ and $\mid\psi\rangle$ are normalised)

$$W_{\phi\overline{\psi}}(q,p) = \int ds\, e^{-ips/\hbar}\,\psi^{*}(q-s/2)\,\phi(q+s/2) \tag{2.7}$$

This Wigner distribution function has been introduced via the standard Weyl formalism [18]. It is thus natural to wonder what is the behaviour with respect to the action of the symplectic group. The symplectic group, in this case $Sp(2)$, acts on the classical phase space (1.42). For $T \in Sp(2)$, the Wigner function gets transformed as:

$$W_{\psi}(T(q,p)) = W_{\hat{U}_{T}(\psi)}(q,p) \tag{2.8}$$

The values of the Wigner function along symplectic orbits is related to the action of the unitary representation of the symplectic (properly metaplectic) group on the Hilbert space of states. This unitary representation is dictated by the Von Neumann theorem, developed in the study of the covariance properties of a standard Weyl system.

A very important aspect of this construction is that Wigner functions can assume negative values in some regions of the classical phase space: this is the reason why they are actually called *quasi-probabilities distributions.* Nevertheless their marginal distributions do reproduce true probability densities:

$$\begin{aligned}\int dp\, W_{\psi}(q,p) &= 2\pi\left|\psi(q)\right|^{2}\\ \int dq\, W_{\psi}(q,p) &= 2\pi\left|\tilde{\psi}(p)\right|^{2}\end{aligned} \tag{2.9}$$

Here $\tilde{\psi}$ is the Fourier transform of the wave function ψ in the usual Hilbert space of square integrable functions on the line. In the dynamics of one dimensional point particle, the modulus square of ψ, and $\tilde{\psi}$, represent, in the Schrödinger realization, the probability distributions in the spectral representation of position and momentum.

In this picture, the Wigner functions can also be written as:

$$W_{\hat{\rho}}(q,p) = \int dxdy\,\delta\left(q-\frac{x+y}{2}\right)\,e^{ip(x-y)/\hbar}\psi^{*}(x)\,\psi(y) \tag{2.10}$$

that is:

$$W_{\hat{\rho}}(q,p) = \int dxdy\,\delta\left(q-\frac{x+y}{2}\right)\,e^{ip(x-y)/\hbar}\langle y\mid\hat{\rho}\mid x\rangle \tag{2.11}$$

while, in the momentum representation:

$$W_{\hat{\rho}}(q,p) = \int dkdl\, \delta\left(p - \frac{k+l}{2}\right) e^{-iq(l-k)/\hbar}\tilde{\psi}^*(l)\,\tilde{\psi}(k) \tag{2.12}$$

or, equivalently:

$$W_{\hat{\rho}}(q,p) = \int dkdl\, \delta\left(p - \frac{l+k}{2}\right) e^{-iq(l-k)/\hbar}\langle k \mid \hat{\rho} \mid l\rangle \tag{2.13}$$

In these relations, $\mid x\rangle$ and $\mid y\rangle$ are generalized eigenstates of the position observable, while $\mid k\rangle$ and $\mid l\rangle$ are generalised eigenstates of the momentum observable.

2.2 From Wigner functions to Weyl map

In the last section, the theory of Wigner "quasi-probabilities" functions has been developed as a nice example in the Weyl formalism. This is not the historical order these concepts were introduced. The method of Wigner distributions as a descriptions of states of quantum mechanical systems appeared in 1932 [49], quite early in the history of quantum mechanics. As it has been outlined, for systems whose kinematics is based upon the Heisenberg canonical commutation relations, it gives a way of describing both pure and mixed states in a classical phase space setting, at the level of density operators. Moreover, they make it possible to write quantum expectation values in terms of statistical averages on the classical phase space of the system, in a formal analogy with the classical statistical approach. In this perspective, the important aspect is that these quantum distributions are not necessarily positive, thus preventing from the complete identification with a classical distribution. It was later appreciated that Wigner distribution approach to quantum states is naturally dual to the Weyl approach to quantum observables (described at lenght in the last chapter) [48]: together with the work of Moyal [28], who introduced the concept of non abelian products in the space of classical functions, these constitute a complete and coherent formulation of quantum mechanics in terms of c-number dynamical phase space variables, well suited for the comparison with classical mechanics.

2.2.1 A Weyl-Wigner map for functions on a cylinder?

The formalism developed up to now is perfectly fitting for the analysis of quantum systems whose classical counterparts can be considered as point particles moving in a cartesian configuration space, whose cotangent bundle is a vector space.

Is it possible to define a Weyl-Wigner formalism for classical systems whose configuration space is a compact simple Lie group?

In the last chapter, the path followed to define a Weyl map for functions on a plane, i.e. the cotangent bundle of a line, has gone through the definition of a Weyl system for the translation group, in terms of the so called Displacement operators. These operators have been considered as a sort of generalized basis for a space of operators: the coefficients of an expansion are given by the Fourier coefficients of functions on the plane. So a Weyl map seems to be traced to the study of harmonic analysis for the group of translations, and unitary representations of abelian groups defined on the dual space of the classical phase space.

As a first example of what are the features, and the problems, of a generalization along these lines, one can consider the case where $\mathcal{Q}$, the configuration space for a classical system, is a circle S^1, that is the group manifold of $U(1)$. The cotangent space of a circle is a cylinder $T^*S^1 \approx S^1 \times \mathbb{R}$. Considering $\mathbb{R}$ as the real additive group in one dimension, the dual of this classical phase space is the product $\mathbb{Z} \times \mathbb{R}$.

A first naive approach would be that of defining a system of Displacement operators for this space.

On the Hilbert space of square integrable functions on a circle $\mathcal{H} = \mathcal{L}^2\left(S^1, d\theta/2\pi\right)$ an orthonormal basis is given by $\phi_n(\theta) = e^{in\theta}$, with $n \in \mathbb{Z}$. Using a ket notation, one can introduce two sets of unitary operators:

$$\begin{array}{lll} \hat{U} : \mathbb{Z} \mapsto \mathcal{U}(\mathcal{H}) & \hat{U}(m) \mid \phi_n\rangle = \mid \phi_{n+m}\rangle & \\ \hat{V} : \mathbb{R} \mapsto \mathcal{U}(\mathcal{H}) & \hat{V}(k) \mid \phi_n\rangle = e^{-ink} \mid \phi_n\rangle & \quad (2.14) \end{array}$$

satisfying a sort of canonical commutation rules:

$$\hat{U}(m)\hat{V}(k) = e^{imk}\hat{V}(k)\hat{U}(m) \tag{2.15}$$

A Displacement operator can be defined:

$$\hat{D}(m,k) = e^{imk/2}\hat{V}(k)\hat{U}(m) = e^{-imk/2}\hat{U}(m)\hat{V}(k) \tag{2.16}$$

This set of operators do form a complete (trace orthonormal) basis for all operators on $\mathcal{H}$.

A natural idea would be mapping an operator to a function via:

$$\tilde{f}(m,k) = Tr\left[\hat{f}\hat{D}^\dagger(m,k)\right] \tag{2.17}$$

At a first sight, this $\tilde{f}$ seems to be a function defined on the dual $\mathbb{Z} \times \mathbb{R}$, eventually to be identified with the Fourier transform of a function on the cylinder. But a deeper inspection says that, for $k - k' = 4\pi$:

$$\tilde{f}(m,k) = \tilde{f}(m,k')$$

This means that $\tilde{f}$ is actually a function on $S^1 \times \mathbb{Z}$. This procedure does not enable to write a Weyl symbol on the "right" classical phase space.

It is however possible to define a different system of operators, the use of which does give a version of a Wigner formalism for this class of quantum systems. The goal will be the introduction of a set of Wigner quasi-probability densities, requiring that their marginals are the correct probability distribution for these systems. Instead of defining a set of Displacement operators, the line will be the definition of a set of Weyl operators, generalizing those in (1.55).

On the same Hilbert space, that of square integrable functions on the circle, with the normalized Haar measure, one can define different pairs of operators:

$$\begin{array}{lll} \hat{U} : \mathbb{Z} \mapsto \mathcal{U}(\mathcal{H}) & \left(\hat{U}(m)\psi\right)(\theta) = e^{im\theta}\psi(\theta) & \\ \hat{V} : S^1 \mapsto \mathcal{U}(\mathcal{H}) & \left(\hat{V}(\theta')\psi\right)(\theta) = \psi([\theta - \theta']) & \quad (2.18) \end{array}$$

here $[\theta - \theta']$ means $(\theta - \theta')\,/mod 2\pi$. $\hat{U}(\cdot)$ is a unitary representation of $(\mathbb{Z}, +)$, while $\hat{V}(\cdot)$ is a unitary representation of $U(1)$.

Their action on the separable orthonormal basis of ket vectors $\mid \phi_n \rangle$ is:

$$\begin{aligned} \hat{U}(m) \mid \phi_n \rangle &= \mid \phi_{n+m} \rangle \\ \hat{V}(\theta) \mid \phi_n \rangle &= e^{-in\theta} \mid \phi_n \rangle \end{aligned} \tag{2.19}$$

and they satisfy an "Heisenberg"-type commutation relation:

$$\hat{V}(\theta)\,\hat{U}(m) = e^{-im\theta}\,\hat{U}(m)\,\hat{V}(\theta) \tag{2.20}$$

Here the phase factor can be properly formalised as a character of the representation $\hat{V}(\cdot)$ of $U(1)$: this notion generalizes the action of covector on a vector in the case of the group of translations. The composition of $\hat{U}$ operators and $\hat{V}$ operators with a suitable phase factors would give the analogue of displacement operators, one for each point of the space $S^1 \times \mathbb{Z}$: even in this case they do form a complete trace orthonormal system:

$$Tr\left[\hat{U}(m)\,\hat{V}(\theta)\,\hat{V}^{\dagger}(\theta')\,\hat{U}^{\dagger}(m')\right] = \delta_{mm'}\,\delta([\theta - \theta']) \tag{2.21}$$

In this relation the continuous $\delta([\theta - \theta'])$ is referred to the measure $d\theta/2\pi$.

In the Hilbert space on which these operators have been realized, besides the separable basis $\phi_n(\theta)$ already mentioned, it is possible to introduce a sort of generalized overcomplete continuum basis of ket states $\mid \theta \rangle$. They are introduced in analogy to eigenstates of the position observable for a quantum point particle in a cartesian space. They are defined by:

$$\phi_n(\theta) = e^{in\theta} = \langle \theta \mid \phi_n \rangle \tag{2.22}$$

Normalization and overcompleteness are written as:

$$\langle \theta \mid \theta' \rangle = \delta([\theta - \theta']) \tag{2.23}$$

$$\mathbf{1} = \int \frac{d\theta}{2\pi} \mid \theta \rangle \langle \theta \mid \tag{2.24}$$

In this basis these operators act as:

$$\begin{aligned} \hat{U}(m) \mid \theta \rangle &= e^{im\theta} \mid \theta \rangle \\ \hat{V}(\theta') \mid \theta \rangle &= \mid [\theta + \theta'] \rangle \end{aligned} \tag{2.25}$$

Looking at this basis, that is related to a spectral decomposition of a position observable for a particle whose configuration space is a circle, one can see that the discrete basis can be seen as made of eigenstates for a momentum operator of that particle: as it is well known, this observable has discrete spectrum. In a group-theoretical approach, this can be seen as result in harmonic analysis for the group $U(1)$. The dual of a compact group is discrete [39]: it is possible to prove that the space of UIRR's for a compact group is labelled by discrete indices.

Using these operators, it is possible to introduce a new set:

$$\hat{W}(\theta, n) = \sum_m \int \frac{d\sigma}{2\pi}\,\hat{U}^{\dagger}(m)\,\hat{V}(\sigma)\; e^{in\sigma} e^{im(\theta + \sigma/2)} \tag{2.26}$$

These operators are hermitian (thus resembling one of the properties of Weyl operators for the cartesian case (1.55)), and form a complete trace orthonormal system. They are a map from the space $S^1 \times \mathbb{Z}$ to unitaries $\mathcal{U}(\mathcal{H})$: this means that the standard procedure associates to an operator $\hat{A} \in Op(\mathcal{H})$ a symbol on that space:

$$A(\theta, n) = Tr\left[\hat{A}\hat{W}^{\dagger}(\theta, n)\right] \tag{2.27}$$

Which are the properties of these symbols? If one considers the symbol of a projector $\rho = |\psi\rangle\langle\psi|$, then one has:

$$W_{\hat{\rho}}(\theta, n) = Tr\left[\hat{\rho}\hat{W}^{\dagger}(\theta, n)\right] = \langle\psi \mid \hat{W}(\theta, n) \mid \psi\rangle \tag{2.28}$$

If one computes the marginal distribution for the symbol $W_{\hat{\rho}}(\theta, n)$, the result is:

$$\begin{aligned} \int \frac{d\theta}{2\pi} W_{\hat{\rho}}(\theta, n) &= \langle\psi \mid \phi_n\rangle\langle\phi_n \mid \psi\rangle \\ \sum_n W_{\hat{\rho}}(\theta, n) &= \langle\psi \mid \theta\rangle\langle\theta \mid \psi\rangle \end{aligned} \tag{2.29}$$

This shows that marginals reproduce the probability density distribution for the quantum dynamics of a point particle on a circle, written in the dual basis of position and momentum. By analogy with the planar case, this is the reason why the maps:

$$\begin{aligned} \hat{A} &= \sum_n \int \frac{d\theta}{2\pi} A(\theta, n)\, \hat{W}(\theta, n) \\ A(\theta, n) &= Tr\left[\hat{A}\hat{W}^{\dagger}(\theta, n)\right] \end{aligned} \tag{2.30}$$

that define an isomorphism between Hilbert-Schmidt operators in the Hilbert space $\mathcal{H}$, and square modulus measurable functions on the space $S^1 \times \mathbb{Z}$ (considering an integration over the continuous θ and a summation over the discrete n), can be defined a Weyl-Wigner isomorphism for a dynamics of the particle on a circle. The most important novelty this formalism shows is that functions related to quantum observables via this isomorphism are not on the classical phase space, the classical cotangent space of the configuration space S^1, but on a sort of quantum cotangent space of S^1, that is the product of the configuration space itself (the group manifold) with the dual space.

Moreover, a relation of the kind of (1.58) is valid:

$$Tr\left[\hat{A}^{\dagger}\hat{B}\right] = \sum_n \int \frac{d\theta}{2\pi} A^*(\theta, n)\, B(\theta, n) \tag{2.31}$$

2.3 The Wigner distributions in the Lie group case

The path followed in the case of the cylinder, the cotangent bundle for the group manifold of $U(1)$, can be generalized to the case of a compact simple Lie group G, of order n.

For a quantum system whose classical counterpart has a configuration space which is such a Lie group, setting the kinematics means defining the commutation relations (the *quantum conditions*, in Dirac's approach) for a set of fundamental observables, and a realization in terms of operators on a suitable Hilbert space. Quantum kinematics for these system shows that noncommutativity among fundamental observables are traced to the nonabelianess of the group G, and not to the symplectic structure as in the cartesian case.

The Wigner distributions are then introduced, attempting to generalize the covariance properties of the Wigner distribution in the cartesian case. The interesting aspect is the definition of a kind of quantum cotangent bundle for a Lie group, related to the space of unitary irreducible representations of the group itself.

Once the general isomorphism is set, it is possible to recover the case of the classical cylinder, used as a guide in the last section, as a particular case.

2.3.1 Classical Kinematics

The classical system under analysis has a configuration space $\mathcal{Q}$ which is a compact simple Lie group G. The corresponding phase space is T^*G, the cotangent bundle of the group. It can be described both in intrinsic geometric terms, and in a local coordinates system [19].

A Lie group is represented by a parallelizable and differentiable manifold. Its Lie algebra $\underline{G}$ can be identified with the tangent space of G at the identity, and is isomorphic to the dual, the cotangent space:

$$\underline{G} = T_e G = T_e^* G = \underline{G}^*$$

The Lie group brings with it the set of left translations L_g and the set of right translations R_g: these are mutually commuting actions of G by mapping of G on itself:

$$\begin{array}{lclr} L_g : G & \mapsto & G \;\; L_g(g') = gg' & \\ R_g : G & \mapsto & G \;\; R_g(g') = g'g^{-1} & \quad (2.32) \end{array}$$

The corresponding tangent maps and pull backs act, as nonsingular linear transformations, on the tangent and cotangent spaces, respectively at general points of G, according to:

$$\begin{array}{rcl} (L_g)_* : T_{g'}G & \mapsto & T_{gg'}G \\ (R_g)_* : T_{g'}G & \mapsto & T_{g'g^{-1}}G \\ (L_g)^* : T^*_{g'}G & \mapsto & T^*_{g^{-1}g'}G \\ (R_g)^* : T^*_{g'}G & \mapsto & T^*_{g'g}G \end{array} \quad (2.33)$$

If dual bases are introduced $\{e_r\}$ for T_eG and $\{e^r\}$ for T_e^*G, then the action of these right and left tangent maps defines two bases for general vector fields on G:

$$\begin{array}{rcl} X_r(g) & = & \left(R_{g^{-1}}\right)_*(e_r) \\ \tilde{X}_r(g) & = & (L_g)_*(-e_r) \end{array} \quad (2.34)$$

Vector fields $\{X_r\}$ are right invariant, and are the generators of the left translations L_g, while the vector fields $\{\tilde{X}_r\}$ are left invariant and generate the right translations R_g. They close a representation of the Lie algebra $\underline{G}$:

$$\begin{aligned} [X_r, X_s] &= c^t_{rs} X_t \\ \left[\tilde{X}_r, \tilde{X}_s\right] &= -c^t_{rs} \tilde{X}_t \\ \left[X_r, \tilde{X}_s\right] &= 0 \end{aligned} \tag{2.35}$$

In general the elements of a Lie group G cannot be described with the help of coordinates in a globally smooth manner. In particular this is so if G is compact: one has to work with charts, with well defined transition rules in overlaps. An element $g \in G$ can be locally labelled by n real independent continuous coordinates q^r. Conventionally it is set $q^r = 0$ at the group identity e. The bases elements for $T_e G$ and $T^*_e G$ are identified with: (here the subscript 0 means that these quantities are evaluated at the group identity)

$$e_r = \left(\frac{\partial}{\partial q^r}\right)_0 \qquad e^r = (dq^r)_0 \tag{2.36}$$

The product of two group elements $g(q)$ and $g'(q')$ is a function:

$$g(q) \cdot g'(q') = (gg')(f(q, q')) \tag{2.37}$$

and, infinitesimally:

$$\begin{aligned} \eta^r_s(q) &= \left(\frac{\partial f^r}{\partial q'^s}\right)(q', q)_{q'=0} \\ \tilde{\eta}^r_s(q) &= \left(\frac{\partial f^r}{\partial q'^s}\right)(q, q')_{q'=0} \end{aligned} \tag{2.38}$$

These quantities are related to the coordinate expression of the left and right invariant vector fields:

$$\begin{aligned} X_r &= \eta^s_r(q) \frac{\partial}{\partial q^s} \\ \tilde{X}_r &= -\tilde{\eta}^s_r(q) \frac{\partial}{\partial q^s} \end{aligned} \tag{2.39}$$

In the sense of classical canonical mechanics, on T^*G there are local canonically conjugate momentum variables p_r, and the classical Poisson bracket relations are:

$$\begin{aligned} \{q^r, q^s\} &= 0 \\ \{q^r, p_s\} &= \delta^r_s \\ \{p_r, p_s\} &= 0 \end{aligned} \tag{2.40}$$

The range of the p_r variables is usually taken to be the entire real line: so to define an identification $T^*_e G \simeq \mathbb{R}^n$ at each $g \in G$. These local coordinates can be used to define a system of global ones, introducing a system of generalised momenta:

$$\begin{aligned} J_s &= \eta^r_s(q)\, p_r \\ \tilde{J}_s &= -\tilde{\eta}^r_s(q)\, p_r \end{aligned} \tag{2.41}$$

At the end of this description, there is the explicit evaluation of the Poisson bracket relations this system satisfies:

$$\begin{aligned} \{q^r, q^s\} &= 0 \\ \{q^r, J_s\} &= \eta^r_s(q) \\ \{J_r, J_s\} &= c^t_{rs} J_t \end{aligned} \tag{2.42}$$

Similar relations do occur for coordinates suited for the right action.

2.3.2 Quantum Kinematics

The most natural Hilbert space on which trying quantizing this class of classical system is the set of complex square integrable functions on G, with respect to the normalized Haar measure $\int_G d\mu = 1$:

$$\mathcal{H} = \mathcal{L}^2(G, d\mu) = \{\psi(g) \in \mathbb{C} : \| \psi \|^2 = \int_G d\mu \, |\psi(g)|^2 < \infty\}$$

As in the previous example, the first thing to do is to define a set of operators from points of the group G to unitary operators in this Hilbert space. They will encode the informations about position observable for the system. From a noncommutative geometry point of view, the notion of position coordinates is intrinsically captured by the commutative algebra of smooth functions $\mathcal{F}(G)$. To a function $f \in \mathcal{F}(G)$ one can associate an operator on $\mathcal{H}$:

$$\left(\hat{f}\psi\right)(g) = f(g)\psi(g) \tag{2.43}$$

The algebra of functions on the group is mapped into an abelian algebra of multiplicative operators. The next step is to define the analogous of momentum observables. They will be related to the generators of the dual space of the group G.

Both left and right actions are defined on this Hilbert space. In particular, the left action is written as:

$$\left(\hat{V}(g')\psi\right)(g) = \psi\left(g'^{-1}g\right) \tag{2.44}$$

These operators define a unitary representation of the group G:

$$\hat{V}(g')\hat{V}(g) = \hat{V}(g'g)$$

So it is possible to identify the Hermitian generators. Fixed the basis e_a in the Lie algebra $\underline{G}$, the exponential map for a compact group is surjective, so every element can be written as:

$$g = \exp(\alpha_r e_r) \tag{2.45}$$

and, once more, Stone's theorem enables to define a set of Hermitian generators for the action represented by:

$$\hat{V}(\exp(\alpha_r e_r)) = \exp\left(-i\alpha_r \hat{J}_r\right) \tag{2.46}$$

These generators do represent the Lie algebra $\underline{G}$ on $\mathcal{H}$, and will be considered as generalized momenta:

$$\left[\hat{J}_a; \hat{J}_b\right] = ic^s_{ab}\hat{J}_s \tag{2.47}$$

On wave functions $\psi(g)$ of this Schrödinger representation, each $\hat{J}_r$ acts a first order partial differential operator, that is a vector field, thus defining another representation of $\underline{G}$ as derivations (X_r are the right invariant vector fields (2.39)):

$$\left(\hat{J}_r\psi\right)(g) = iX_r\psi(g) \tag{2.48}$$

It is possible to express functions of position also via unitary operators, to be as close as possible to the Weyl approach. For a real function in $\mathcal{F}(G)$, a unitary operator is given by:

$$\hat{U}(f) = e^{i\hat{f}} \qquad \left(\hat{U}(f)\psi\right)(g) = e^{if(g)}\psi(g)$$

It can be seen that they satisfy a relation:

$$\left(\hat{U}(f)\hat{V}(g')\psi\right)(g) = e^{i\left[f(g)-f\left(g'^{-1}g\right)\right]}\left(\hat{V}(g')\hat{U}(f)\psi\right)(g) \tag{2.49}$$

which is in the spirit of (1.21), except that f is not restricted to be linear in any coordinate variables.

As in the previous case, two bases for the Hilbert space at hand can be introduced. The first is a "momentum" basis, and can be set up using the Peter-Weyl theorem involving the unitary irreducible representations of G. The various UIR's can be denoted by an index j, in general a collection labelling the Casimir operators eigenvalues for the group. Since G is compact, every UIR is finite dimensional, and its dimension is N_j. Rows and columns within the j^{th} representation are labelled by m and n, a sort of generalised magnetic quantum numbers. So a unitary matrix representing the element $g \in G$ in the j^{th} UIR is:

$$g \mapsto \left(D^j_{mn}(g)\right) \tag{2.50}$$

Moreover, there is a freedom of unitary changes in the choice of m, n. Unitarity and associativity of the representations are written as:

$$\begin{aligned}\sum_n D^j_{mn}(g)^* D^j_{m'n}(g) &= \delta_{mm'} \\ \sum_n D^j_{mn}(g') D^j_{nn'}(g) &= D^j_{mn'}(g'g)\end{aligned} \tag{2.51}$$

while orthogonality and completeness as:

$$\begin{aligned}\int_G d\mu\, D^j_{mn}(g) D^{j'}_{m'n'}(g)^* &= \delta_{jj'}\delta_{mm'}\delta_{nn'}/N_j \\ \sum_{jmn} N_j D^j_{mn}(g) D^j_{mn}(g')^* &= \delta\left(g^{-1}g'\right)\end{aligned} \tag{2.52}$$

This completeness property, expressed by the presence of a distributional δ, is the main result of the mentioned Peter-Weyl theorem. It enables to perform an harmonic analysis, generalizing the Fourier analysis on linear spaces:

$$\begin{aligned}f(g) &= \sum_{jmn} f^j_{nm} D^j_{nm}(g)\sqrt{N_j} \\ f^j_{nm} &= \int_G d\mu\, \left(D^j_{nm}(g)\right)^* f(g)\sqrt{N_j}\end{aligned} \tag{2.53}$$

This analysis shows that a basis in $\mathcal{H}$ is given by, in ket notation, vectors $\mid jmn\rangle$, for which:

$$\begin{aligned} \langle j'm'n' \mid jmn\rangle &= \delta_{jj'}\delta_{mm'}\delta_{nn'} \\ \hat{V}(g) \mid jmn\rangle &= \sum_{m'} D^{j}_{mm'}\left(g^{-1}\right) \mid jm'n\rangle \end{aligned} \tag{2.54}$$

The action of the operator $\hat{V}(g)$ on this basis shows a well known fact. Left regular representation of a compact group is not irreducible, and the multiplicity of occurrence of the j^{th} UIR in its reduction is equal to the dimension N_j of the irreducibility subspace. The index n counts this multiplicity [39].

The second basis is related to "position" observables. For each element of the group G, represented by a point g on the manifold, it is possible to set:

$$\langle jmn \mid g\rangle = \sqrt{N_j}\left(D^{j}_{mn}(g)\right)^{*} \tag{2.55}$$

These vectors define an orthonormal, in a generalized sense with respect to the Haar measure, and overcomplete system in $\mathcal{H}$:

$$\langle g \mid g'\rangle = \delta\left(g^{-1}g'\right) \qquad \mathbf{1} = \int_G d\mu \mid g\rangle\langle g \mid \tag{2.56}$$

$\hat{V}(\cdot)$, the left representation, acts as:

$$\hat{V}(g') \mid g\rangle = \mid g'g\rangle \tag{2.57}$$

so that it is the representation of the left multiplication for element of the group.

In this context, the right action of G can be defined as:

$$\begin{aligned} \hat{R}(g') \mid g\rangle &= \mid gg'^{-1}\rangle \\ \hat{R}(g') \mid jmn\rangle &= \sum_{n'} D^{j}_{n'n}(g') \mid jmn'\rangle \end{aligned} \tag{2.58}$$

It is clear that, for right regular representation of G on this space, it is the index m that counts the multiplicity of the occurrence of the j^{th} UIR in its reduction.

Given a state $\mid \psi\rangle$ in $\mathcal{H}$, it can be expanded in these two basis:

$$\begin{aligned} \langle g \mid \psi\rangle &= \psi(g) \\ \langle jmn \mid \psi\rangle &= \psi_{jmn} = \sqrt{N_j}\int_G d\mu \left(D^{j}_{mn}(g)\right)^{*}\psi(g) \\ \parallel \psi \parallel^2 &= \sum_{jmn} \left|\psi_{jmn}\right|^2 = \int_G d\mu \left|\psi(g)\right|^2 \end{aligned} \tag{2.59}$$

2.3.3 A Wigner distribution

The analysis of the cartesian case, and of the examples above, might give rise to the hypothesis that, for a state $\mid \psi\rangle$, the corresponding Wigner function W were a function of arguments g, bilinear in ψ, (more precisely involving one ψ factor and one ψ^* factor) and JMN (quantised momenta), such that integration over g yields $\left|\psi_{JMN}\right|^2$ while summation over JMN yields $\left|\psi(g)\right|^2$. This would be a natural way in which the marginal distributions are reproduced.

A natural requirement of covariance of this distribution for both the right and left actions of G on state ψ poses the problem of choosing the arguments of $W(\ldots)$ in such a way to allow for a natural linear transformation law under each of the changes $\psi(g) \rightarrow \psi\left(g'^{-1}g\right)$ (left action) and $\psi(g) \rightarrow \psi(gg')$ (right action) on ψ. In particular, eqs.(2.54) and (2.58) show that, in the discrete JMN basis, one index carries the transformation properties for the left action, while the other carries for the right action. Since W should involve a bilinear expression of the kind $\psi\psi^*$, it becomes a reasonable assumption that a Wigner distribution function for this system is a function:

$$\psi(g) \;\mapsto\; \tilde{W}(g; JMN\, M'N')$$

This means that, from this point of view, W is a complex function defined on a space which is the product of G itself times a lattice, related to the set of UIR's for the group G. Moreover, this analysis does not appear to be necessary in the cartesian case and in the previous $G = U(1)$ case, as they are abelian. In that case left and right actions are the same.

The properties this distribution should satisfy are[1]:

- complex conjugation exchanges primed with unprimed indices (this assures the expected transformation property for hermitian conjugation when the Wigner distribution represents a density operator):

$$\left(\tilde{W}(g; JMN\, M'N')\right)^* = \tilde{W}(g; JM'N'\, MN); \tag{2.60}$$

- it reproduces the expected marginal distributions:

$$\begin{aligned} \int_G d\mu\, \tilde{W}(g; JMN\, MN) &= \left|\psi_{JMN}\right|^2 \\ \sum_{JMN} \tilde{W}(g; JMN\, MN) &= \left|\psi(g)\right|^2 \end{aligned} \tag{2.61}$$

- for a left action of G the transformation of the state $\psi'(g) = \psi\left(g'^{-1}g\right) \rightarrow$

$$\tilde{W}'(g; JMN\, M'N') = \sum_{M_1 M_1'} D^J_{MM_1}(g')\, D^J_{M'M_1'}(g')^*\, \tilde{W}\left(g'^{-1}g; JM_1N\, M_1'N'\right) \tag{2.62}$$

- for a right action of G the transformation of the state $\psi'(g) = \psi(gg') \rightarrow$

$$\tilde{W}''(g; JMN\, M'N') = \sum_{N_1 N_1'} D^J_{N_1 N}\left(g'^{-1}\right) D^J_{N_1'N'}\left(g'^{-1}\right)^* \tilde{W}(gg'; JMN_1\, M'N_1') \tag{2.63}$$

Eq.(2.10) suggests that a form of Wigner function is:

$$\tilde{W}(g; JMN\, M'N') = N_J \int_G d\mu' \int_G d\mu''\, \delta\left(g^{-1}s(g', g'')\right)\, D^J_{MN}(g')\, \psi^*(g')\, D^J_{M'N'}(g'')^*\, \psi(g'') \tag{2.64}$$

[1] It can be checked that these properties are compatible among themselves.

The expression $s(g', g'')$ is a group element, depending on the two variables g', g''. It is a generalization of the average element $\frac{x+y}{2}$ in the cartesian case. Without entering into a characterization of this function, it can be seen that if $s(g', g'')$ is the midpoint along the geodesic curve (with respect to the invariant Cartan-Killing metric on G) joining g' to g'' then the eq.(2.64) defines an acceptable Wigner distribution for a quantum mechanical system on a compact semisimple Lie group [2][30].

2.3.4 A Weyl-Wigner isomorphism

The definition (2.64) can be immediately extended to associate a function $\tilde{W}_{\hat{A}}(g; jmn\, m'n')$, a symbol, to every linear operator $\hat{A}$ on $\mathcal{H}$ of Hilbert-Schmidt class. In terms of the integral kernel $\langle g'' \mid \hat{A} \mid g' \rangle$ of $\hat{A}$, in a way similar to eq.(2.11), one has:

$$\tilde{W}_{\hat{A}}(g; jmn\, m'n') = N_j \int_G d\mu' \int_G d\mu''\, \delta\left(g^{-1} s(g'g'')\right) D^j_{m'n'}(g'')^* D^j_{mn}(g') \langle g'' \mid \hat{A} \mid g' \rangle \tag{2.66}$$

It is the case that this expression determines $\hat{A}$ completely, however this happens in an overcomplete manner: there are certain linear relations obeyed by $\tilde{W}_{\hat{A}}$ which have an $\hat{A}$ independent form. Properties of the function $s(g', g'')$ make it possible to prove that, for this symbol:

$$\begin{aligned} &\sum_{m'n'} D^j_{m'm''}(g)\, D^j_{n''n'}(g)\, \tilde{W}_{\hat{A}}(g; jmn\, m'n') = \\ = \;& N_j \int_G d\mu' \int_G d\mu''\, \delta\left(g^{-1} s(g', g'')\right) \langle g g'^{-1} g \mid \hat{A} \mid g' \rangle D^j_{n''m''}(g')\, D^j_{mn}(g') \end{aligned} \tag{2.67}$$

The r.h.s. of this relation shows a symmetry under the simultaneous interchanges $m \leftrightarrow n''$ and $n \leftrightarrow m''$, and this is independent of $\hat{A}$: so l.h.s. should have this symmetry too. This is the sense in which $\tilde{W}_{\hat{A}}(g; jmn\, m'n')$ contains information about $\hat{A}$ in an overcomplete manner, and this happens when G is non abelian. Taking advantage of this, one can associate a symbol to an operator in a simpler way:

$$\begin{aligned} A(g, jmm') &= N_j^{-1} \sum_n \tilde{W}_{\hat{A}}(g, jmn\, m'n) \\ &= \int_G d\mu' \int_G d\mu''\, \delta\left(g^{-1} s(g', g'')\right) \langle g'' \mid \hat{A} \mid g' \rangle D^j_{mm'}\left(g' g''^{-1}\right) \end{aligned} \tag{2.68}$$

This relation can be compared to the cartesian case (2.11). In the δ factor the role of $\frac{x+y}{2}$ is played by the geodesic average $s(g', g'')$; position eigenstates are now $\mid g' \rangle$ and $\mid g'' \rangle$ and the plane wave factor is now a unitary irreducible

[2] The geodesic curve is the solution to the variational problem:

$$\delta \int_{\sigma_1}^{\sigma_2} d\sigma \left[g_{rs}(q(\sigma)) \frac{dq^r}{d\sigma} \frac{dq^s}{d\sigma} \right]^{1/2} = 0 \tag{2.65}$$

where g_{rs} are the components of the Riemannian metric tensor whose value in the identity of G is given in terms of the structure constants of the Lie algebra $\underline{G}$, $g_{rs}(e) = -c^v_{ru} c^u_{sv}$, and whose value in a generic point of is obtained by shifting that of the origin acting with left and right shift (translations).

representation of the group G. For example, if $\hat{A}$ is a multiplication operator:

$$\hat{A} = \int_G d\mu\, f(g)\ |\, g\rangle\langle g\,| \tag{2.69}$$

then its symbol is:

$$A(g, jmn) = f(g)\,\delta_{mn} \tag{2.70}$$

This is defined the Weyl symbol corresponding to the operator $\hat{A}$, and it is very close to expression (2.11). The passage $\hat{A} \rightarrow \hat{A}^{\dagger}$ results in:

$$A^{\dagger}(g, jmm') = A(g, jm'm)^{*} \tag{2.71}$$

The transformation properties of the Weyl symbol under a left or right regular representation of G:

$$\begin{aligned} \hat{A}' &= \hat{V}(g')\,\hat{A}\hat{V}^{\dagger}(g') \rightarrow \\ A'(g; jmm') &= \sum_{m_1 m_1'} D^j_{mm_1}(g')\, D^j_{m'm_1'}(g')^{*}\ A\left(g'^{-1}g; jm_1m_1'\right) \end{aligned} \tag{2.72}$$

and:

$$\begin{aligned} \hat{A}'' &= \hat{R}(g')\,\hat{A}\hat{R}^{\dagger}(g') \rightarrow \\ A''(g; jmm') &= A(gg'; jmm') \end{aligned} \tag{2.73}$$

Had one chosen, in (2.68), to sum over the other pair of indices, the symbol would have had these covariance properties interchanged. Another very interesting relation these symbols satisfy is:

$$Tr\left[\hat{A}\hat{B}\right] = \sum_{jmn} N_j \int_G d\mu\, A(g, jmn)\, B(g; jnm) \tag{2.74}$$

The very important aspect of the formalism one has developed is that it can be cast in the form of a Weyl-Wigner isomorphism, in a way formally similar to (1.56), introducing a set of quantizer operators.

It can be seen that the symbol can be written as:

$$A(g; jmn) = Tr\left[\hat{A}\hat{W}(g, jmn)\right] \tag{2.75}$$

where the quantizer operators are given by:

$$\hat{W}(g, jmn) = \sum_{j'm'n'} N_{j'} \int_G d\mu'\, \hat{U}(j'n'm')\, \hat{V}(g')\ D^j_{mn}(g')\, D^{j'}_{n'm'}\left(g^{-1}s_0\left(g'^{-1}\right)\right) \tag{2.76}$$

In this expression, operator $\hat{U}(\cdot)$ is defined by:

$$\left(\hat{U}(jmn)\,\psi\right)(g) = D^j_{mn}(g)\,\psi(g) \tag{2.77}$$

They are analogous to the operators $\hat{U}(m)$ defined by eq.(2.18) for the case of the group $U(1)$: the difference with those is that they are not unitary. Nevertheless they satisfy a related condition:

$$\sum_M \hat{U}^{\dagger}(jMn)\,\hat{U}(jMn') = \sum_M \hat{U}^{\dagger}(jnM)\,\hat{U}(jn'M) = \delta_{n'n}\mathbf{1} \tag{2.78}$$

The analogy of this operator $\hat{W}(g, jmn)$ (2.76) with the quantizer used in the cartesian case is evident, as $s_0(g)$ is a shorthand for $s(e,g)$, the midpoint in the geodesic joining the point g to the identity. Exponential map for compact groups enables to recover:

$$s_0(g) \;=\; s_0(\exp(\alpha_r e_r)) \;=\; s_0\left(\exp\left(\frac{1}{2}\alpha_r e_r\right)\right) \tag{2.79}$$

These quantizers satisfy:

$$\hat{W}^{\dagger}(g; jmn) \;=\; \hat{W}(g; jnm) \tag{2.80}$$

and are a complete trace orthonormal system:

$$Tr\left[\hat{W}^{\dagger}(g'; j'm'n')\,\hat{W}(g; jmn)\right] \;=\; N_j^{-1}\delta_{jj'}\delta_{mm'}\delta_{nn'}\delta\left(g^{-1}g'\right) \tag{2.81}$$

so that a Weyl map, a quantization map, can be defined as:

$$\hat{A} \;=\; \sum_{jmn} N_j \int_G d\mu\, A(g; jmn)\,\hat{W}(g; jnm) \tag{2.82}$$

This relation establish one of the map in a Weyl-Wigner isomorphism built starting from a quantum system on a compact simple Lie group: This isomorphism maps Hilbert-Schmidt operators into functions in the space $\mathcal{F}(G \times \Gamma)$ where Γ is a lattice, a set of discrete indices related to unitary irreducible representations of the group G. This space, in analogy with the exemplum case of $G = U(1)$, is a sort of quantum cotangent space.

2.3.5 A noncommutative product among functions on a Quantum Cotangent Space

The isomorphism just outlined enables in a natural way to define a non abelian product on the set of functions on what has been called a quantum cotangent space:

$$(A * B)(g; \gamma) \;=\; Tr\left[\hat{A}\hat{B}\hat{W}(g, \gamma)\right] \tag{2.83}$$

(here $(g;\gamma)$ is a short cut for $(g; jmn)$ while $(g; \underline{\gamma})$ stands for $(g; jnm)$- as it has been seen, the ordering of the labels is important -).

Hence:

$$(A * B)(g, \gamma) = \sum_{\tilde{\gamma}\,\check{\gamma}} N_{\tilde{j}} N_{\check{j}} \int_G d\tilde{g} \int_G d\check{g}\, A(\tilde{g}, \tilde{\gamma})\, B(\check{g}, \check{\gamma}) \left[Tr\, \hat{W}(g, \gamma)\, \hat{W}(\tilde{g}, \underline{\tilde{\gamma}})\, \hat{W}(\check{g}, \underline{\check{\gamma}})\right] \tag{2.84}$$

The product is non local, and the integral kernel is given by the trace term between square brackets. To analyse this term, the first step is to study the possibility of a kind of inversion of (2.76).

Eventually[3]:

$$\hat{U}(\tilde{j}\tilde{n}\tilde{m})\,\hat{V}(\tilde{g}) = \sum_{\gamma} \int_G dg\, \hat{W}(g, \gamma)\, D^{\tilde{j}}_{\tilde{n}\tilde{m}}(s_0(\tilde{g})\, g)\left(D^{j}_{mn}(\tilde{g})\right)^{*} N_j \tag{2.85}$$

[3] In appendix there are the details of the calculation related to this section.

This can be seen as a sort of antitransform of (2.76).

The second step of the analysis just gives the composition properties of $\hat{U}$and $\hat{V}$ operators:

$$\hat{U}\left(j'n'm'\right)\hat{U}\left(j''n''m''\right)=\sum_{JNM,\lambda}C^{j',j'',J\,\lambda}_{n'm',n''m'',NM}\hat{U}\left(JNM\right) \tag{2.86}$$

$$\hat{U}\left(j'n'm'\right)\hat{V}\left(g'\right)\hat{U}\left(j''n''m''\right)\hat{V}\left(g''\right)=\sum_{k=1}^{N_{j''}}\sum_{JNM,\lambda}D^{j''}_{n''k}\left(g'^{-1}\right)C^{j',j'',J\,\lambda}_{n'm',km'',NM}\hat{U}\left(JNM\right)\hat{V}\left(g'g''\right) \tag{2.87}$$

This notation writes in a compact form the product of two UIR's in terms of direct sum of UIR's [39]. The index λ keeps track of multiple occurrences of a given D^{j}[4].

The third step is to study the composition properties in the set of $\hat{W}$. It can be seen that:

$$\begin{aligned}\hat{W}\left(g,\gamma\right)\hat{W}\left(\tilde{g},\tilde{\gamma}\right) &= \sum_{\gamma'\,\gamma''\,\gamma'''}N_{j'}N_{j''}N_{j'''}\int_G dg'\int_G dg''\int_G dg'''\sum_{k=1}^{N_{j''}}\sum_{JNM,\lambda}\cdot D^{j''}_{n''k}\left(g'^{-1}\right)\\ &\cdot\ C^{j',j'',J\,\lambda}_{n'm',km'',NM}\hat{W}\left(g''',\gamma'''\right)D^{J}_{NM}\left(s_o\left(g'g''\right)g'''\right)\left(D^{j'''}_{m'''n'''}\left(g'g''\right)\right)^{*}\cdot\\ &\cdot\ D^{j}_{mn}\left(g'\right)D^{j'}_{m'n'}\left(g^{-1}s_o\left(g'^{-1}\right)\right)D^{\tilde{j}}_{\tilde{m}\tilde{n}}\left(g''\right)D^{j''}_{m''n''}\left(\tilde{g}^{-1}s_o\left(g''^{-1}\right)\right)\end{aligned} \tag{2.89}$$

The definition (2.84) indicates that the problem is evaluating the trace of the product of three $\hat{W}$ operators:

$$\begin{aligned}Tr\left[\hat{W}\left(g,\gamma\right)\hat{W}\left(\tilde{g},\tilde{\gamma}\right)\hat{W}\left(\check{g},\check{\gamma}\right)\right] &= \sum_{\gamma'\,\gamma''\,\gamma'''}N_{j'}N_{j''}N_{j'''}\int_G dg'\int_G dg''\int_G dg'''\sum_{k=1}^{N_{j''}}\sum_{JNM,\lambda}\cdot\\ &\cdot\ D^{j''}_{n''k}\left(g'^{-1}\right)C^{j',j'',J\,\lambda}_{n'm',km'',NM}D^{J}_{NM}\left(s_o\left(g'g''\right)g'''\right)\cdot\\ &\cdot\ \left(D^{j'''}_{m'''n'''}\left(g'g''\right)\right)^{*}D^{j}_{mn}\left(g'\right)D^{j'}_{m'n'}\left(g^{-1}s_o\left(g'^{-1}\right)\right)\cdot\\ &\cdot\ D^{\tilde{j}}_{\tilde{m}\tilde{n}}\left(g''\right)D^{j''}_{m''n''}\left(\tilde{g}^{-1}s_o\left(g''^{-1}\right)\right)\cdot\\ &\cdot\ Tr\left[\hat{W}\left(g''',\gamma'''\right)\hat{W}\left(\check{g},\check{\gamma}\right)\right]\end{aligned} \tag{2.90}$$

Now from trace evaluation of the composition of two quantizer operators, one has for the integral kernel of the star product (2.84):

$$\begin{aligned}Tr\left[\hat{W}\left(g,\gamma\right)\hat{W}\left(\tilde{g},\underline{\tilde{\gamma}}\right)\hat{W}\left(\check{g},\underline{\check{\gamma}}\right)\right] &= \sum_{\gamma'\,\gamma''}\sum_{\Gamma,\lambda}\int_G dg'\int_G dg''\sum_{k=1}^{N_{j''}}N_{j'}N_{j''}D^{j''}_{n''k}\left(g'^{-1}\right)\cdot\\ &\cdot\ C^{j',j'',J\,\lambda}_{n'm',km'',NM}D^{J}_{NM}\left(s_o\left(g'g''\right)\check{g}\right)\left(D^{\check{j}}_{\check{m}\check{n}}\left(g'g''\right)\right)^{*}\cdot\\ &\cdot\ D^{j}_{mn}\left(g'\right)D^{j'}_{m'n'}\left(g^{-1}s_o\left(g'^{-1}\right)\right)D^{\tilde{j}}_{\tilde{n}\tilde{m}}\left(g''\right)\cdot\\ &\cdot\ D^{j''}_{m''n''}\left(\tilde{g}^{-1}s_o\left(g''^{-1}\right)\right)\end{aligned} \tag{2.91}$$

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[4]In the case of the group $SU\left(2\right)$, in terms of the standard Clebsh-Gordan coefficients, this would be:

$$C^{j',j',J}_{n'm',n''m'',NM}=c^{JN}_{j'n',j''n''}c^{JM}_{j'm',j''m''} \tag{2.88}$$

(here Γ is a short for (J, N, M)) The product will be explicitly obtained putting this last equation into (2.84)

There is also another way to evaluate the expression (2.90), and it is based on:

$$\langle g \mid \hat{W}(\tilde{g}, \tilde{\gamma}) \mid g' \rangle = D^{\tilde{j}}_{\tilde{m}\tilde{n}}\left(g g'^{-1}\right) \delta\left(\tilde{g}^{-1} s\left(g g'\right)\right) \tag{2.92}$$

So:

$$\begin{aligned} Tr\left[\hat{W}(g,\gamma)\hat{W}(\tilde{g},\tilde{\gamma})\hat{W}(\breve{g},\breve{\gamma})\right] &= \int_G dg \int_G dg' \int_G dg'' \cdot \\ &\cdot \; D^{j}_{mn}\left(g' g''^{-1}\right) D^{\tilde{j}}_{\tilde{m}\tilde{n}}\left(g'' g'''^{-1}\right) D^{\breve{j}}_{\breve{m}\breve{n}}\left(g''' g^{-1}\right) \cdot \\ &\cdot \; \delta\left(g^{-1} s\left(g' g''\right)\right) \delta\left(\tilde{g}^{-1} s\left(g'' g'''\right)\right) \delta\left(\breve{g}^{-1} s\left(g''' g\right)\right) \end{aligned}$$

2.3.6 Recovering the case $G = U(1)$

In this section it will be shown how these calculations look like in the special case where G is the group $U(1)$. As already said, the group has the manifold structure of a circle S^1: θ is the "coordinate" on this space. The Haar measure is chosen to be normalized:

$$\int_G d\mu = \int_{S^1} \frac{d\theta}{2\pi} = 1 \tag{2.93}$$

The Hilbert space is $\mathcal{H} = \mathcal{L}^2\left(S^1, \frac{d\theta}{2\pi}\right)$, and an orthonormal basis is the set $\phi_n(\theta) = e^{in\theta}$ $(n \in \mathbb{Z})$. Discrete Fourier transform and anti-transform can be summarized in the formula:

$$\sum_{n=-\infty}^{\infty} e^{-in\left(\theta-\theta'\right)} = \delta\left(\theta-\theta'\right) \tag{2.94}$$

An overcomplete unnormalizable basis is given by $\mid \theta\rangle$ such that (eq.2.22)

$$\langle \theta \mid \psi \rangle = \psi(\theta)$$

$$\langle \theta \mid \phi_n \rangle = e^{in\theta}$$

This group is abelian: unitary irreducible representations are one-dimensional. They will be labelled by an integer n: the D functions of the preceding section become simply number (the so called character), so $\hat{U}(\cdot)$ and $\hat{V}(\cdot)$ are:

$$\hat{V}(\theta) \mid \phi_n \rangle = e^{-in\theta} \mid \phi_n \rangle \tag{2.95}$$

$$\hat{U}(m) \mid \phi_n \rangle = \mid \phi_{n+m} \rangle \tag{2.96}$$

that are exactly the operators ad hoc introduced in (2.19). The quantizer (2.76) acquires the specific form:

$$\hat{W}(\theta, n) = \sum_{m'} \int_{S^1} \frac{d\theta'}{2\pi} \hat{U}(m') \hat{V}(\theta') e^{in\theta'} e^{-im'\left(\theta+\theta'/2\right)} \tag{2.97}$$

which coincides with operator (2.26). Now, as before, the first problem is to "invert" this relation:

$$\int_{S^1} \frac{d\theta}{2\pi} \hat{W}(\theta, n) e^{ik\theta} = \sum_{m'} \int_{S^1} \frac{d\theta}{2\pi} \int_{S^1} \frac{d\theta'}{2\pi} \hat{U}(m') \hat{V}(\theta') e^{in\theta'} e^{-im'\left(\theta+\theta'/2\right)} e^{ik\theta} \tag{2.98}$$

Using (2.94) enables to simplify the r.h.s.:

$$\int_{S^1} \frac{d\theta}{2\pi} \hat{W}(\theta, n) e^{ik\theta} = \int_{S^1} \frac{d\theta'}{2\pi} \hat{U}(k') \hat{V}(\theta') e^{in\theta'} e^{-ik\theta'/2} \tag{2.99}$$

Again:

$$\sum_n e^{-in\theta''} \int_{S^1} \frac{d\theta}{2\pi} \hat{W}(\theta, n) e^{ik\theta} = \hat{U}(k) \hat{V}(\theta'') e^{-ik\theta''/2} \tag{2.100}$$

so that:

$$\hat{U}(n) \hat{V}(\theta) = e^{in\theta/2} \sum_m e^{-im\theta} \int_{S^1} \frac{d\theta'}{2\pi} \hat{W}(\theta', m) e^{in\theta'} \tag{2.101}$$

This is the actual form of (2.85) for this example. The second step is the study of composition properties of two of these operators W:

$$\begin{aligned} \hat{W}(\theta, n) \hat{W}\left(\tilde{\theta}, \tilde{n}\right) &= \sum_{n' n''} \int_{S^1} \frac{d\theta'}{2\pi} \int_{S^1} \frac{d\theta''}{2\pi} \hat{U}(n') \hat{V}(\theta') \hat{U}(n'') \hat{V}(\theta'') \cdot \\ &\cdot \; e^{in\theta'} e^{i\tilde{n}\theta''} e^{-in'\left(\theta+\theta'/2\right)} e^{-in''\left(\tilde{\theta}+\theta''/2\right)} \end{aligned} \tag{2.102}$$

One can finally computes:

$$\begin{aligned} \hat{W}(\theta, n) \hat{W}\left(\tilde{\theta}, \tilde{n}\right) &= \sum_{n' n'' n'''} \int_{S^1} \frac{d\theta'}{2\pi} \int_{S^1} \frac{d\theta''}{2\pi} \int_{S^1} \frac{d\theta'''}{2\pi} \hat{W}(\theta''', n''') e^{i\left(n'\theta''-n''\theta'\right)/2} \cdot \\ &\cdot \; e^{-i\left[n'''\left(\theta'+\theta''\right)-\left(n'+n''\right)\theta'''\right]} e^{i\left(n\theta'+\tilde{n}\theta''\right)} e^{-i\left(n'\theta+n''\tilde{\theta}\right)} \end{aligned} \tag{2.103}$$

This relation clearly indicates that the origin of the properties of this formalism should be addressed to the specific form of the commutation relations. Every phase factor in the integral can be written as a skewsymmetric combination of two variables.

The third step is to evaluate the trace. Noting that:

$$Tr\left[\hat{W}(\theta, n) \hat{W}\left(\tilde{\theta}, \tilde{n}\right)\right] = \delta_{n\tilde{n}} \delta\left(\theta - \tilde{\theta}\right) \tag{2.104}$$

one obtains:

$$\begin{aligned} Tr\left[\hat{W}(\theta, n) \hat{W}\left(\tilde{\theta}, \tilde{n}\right) \hat{W}\left(\check{\theta}, \check{n}\right)\right] &= \sum_{n' n''} \int_{S^1} \frac{d\theta'}{2\pi} \int_{S^1} \frac{d\theta''}{2\pi} e^{i\left(n'\theta''-n''\theta'\right)/2} \cdot e^{i\left(\tilde{n}\theta''-n''\check{\theta}\right)} \\ &\cdot \; e^{-i\left[\check{n}\left(\theta'+\theta''\right)-\left(n'+n''\right)\check{\theta}\right]} e^{i\left(n\theta'-n'\theta\right)} \end{aligned} \tag{2.105}$$

Given two symbols, the product induced by this isomorphism is:

$$(A * B)(\theta, n) = \sum_{\tilde{n}\, \check{n}} \int_{S^1} \frac{d\tilde{\theta}}{2\pi} \int_{S^1} \frac{d\check{\theta}}{2\pi} A\left(\tilde{\theta}, \tilde{n}\right) B\left(\check{\theta}, \check{n}\right) Tr\left[\hat{W}(\theta, n) \hat{W}\left(\tilde{\theta}, \tilde{n}\right) \hat{W}\left(\check{\theta}, \check{n}\right)\right] \tag{2.106}$$

2.4 A noncommutative product on the classical cotangent space

As it has been stressed, the formalism outlined defines a Weyl symbol for a certain class of quantum systems as a function $A(g; jmn)$, not on the classical phase space. The nature of these three indices can be once more analysed via the Peter-Weyl theorem. Index j labels the irreducible representations $D^j(\cdot)$: each of these is realised on a finite dimensional Hilbert space, whose elements are labelled by the index m. The left regular representation, which is one of the building blocks of this construction, is highly reducible on the Hilbert space of square integrable functions on the group G. Exactly, the degeneracy of its occurence is equal to N_j, the dimension of the space of representation $\mathcal{H}_j$. This occurrence is taken into account by the index n. The form of the symbol suggests that it can be considered as a complex function defined on the product $G \times \mathcal{H}_0$ of the group with a smaller Hilbert space, namely that carrying each UIR D^j just once.

If $\mathcal{H}_j$ is the linear span $Sp\{| j, m\rangle\}$ of dimension N_j, with $\langle j'm' \mid jm\rangle = \delta_{jj'}\delta_{mm'}$, then this new space is:

$$\mathcal{H}_0 = \sum_j \oplus \mathcal{H}_j \tag{2.107}$$

In other words, a Weyl symbol can be regarded as a function of G tensor a matrix on $\mathcal{H}_0$, with the crucial property that this matrix is block diagonal with respect to the decomposition of $\mathcal{H}_0$ in terms of $\mathcal{H}_j$. Of course, this interpretation is valid only for non abelian groups: the space on which symbols are defined can be seen as a quantum cotangent space of G.

A symbol can be mapped into a block diagonal, g dependent operator on $\mathcal{H}_0$:

$$\underline{\hat{A}} = \sum_{jmn} \sqrt{N_j}\, A(g; jmn) \mid jm\rangle\langle jn \mid \tag{2.108}$$

satisfying:

$$Tr_{\mathcal{H}}\left[\hat{A}\hat{B}\right] = \int_G d\mu\, Tr_{\mathcal{H}_0}\left[\underline{\hat{A}}\,\underline{\hat{B}}\right] \tag{2.109}$$

This operator can thus be written as a sum:

$$\begin{aligned} \underline{\hat{A}}(g) &= \sum_j \oplus \underline{\hat{A}}_j(g) \\ \underline{\hat{A}}_j(g) &= \sum_{m,n} \sqrt{N_j}\, A(g; jmn) \mid jm\rangle\langle jn \mid \end{aligned} \tag{2.110}$$

of terms acting on each irreducibility subspace. On each of these subspaces there is a set of $\hat{J}_r^{(j)}$, Hermitian generators of the left action of G. In this context irreducibility means that it is possible to expand every operator acting on $\mathcal{H}_j$ as a sum of symmetrised polynomials in these variables:

$$\underline{\hat{A}}_j(g) = \sum_{N=0}^{N(j)} \sum_{r_1, r_2, \ldots, r_N} a_{r_1, r_2, \ldots, r_N}(g; j)\, \{\hat{J}_{r_1}^{(j)} \hat{J}_{r_2}^{(j)} \ldots \hat{J}_{r_N}^{(j)}\}_S \tag{2.111}$$

where the symmetrised sum is:

$$\{\hat{J}^{(j)}_{r_1}\hat{J}^{(j)}_{r_2}\dots\hat{J}^{(j)}_{r_N}\}_S = \frac{1}{N!}\sum_{P\in S_N}\left(\hat{J}^{j}_{r_{P(1)}}\dots\hat{J}^{j}_{r_{P(N)}}\right) \tag{2.112}$$

In these relations, the upper limit in the summation over N is determined by the specific UIR D^j; S_N is the permutation group on N elements, the superscript j denotes the specific realization in the subspace $\mathcal{H}_j$. The coefficients $a_{r_1,\dots,r_N}(g;j)$ are c-numbers quantities symmetric in $r_1,\dots,r_N$. Their dependence by j can be replaced by a dependence on the mutually commuting Casimir operators $\hat{C}$, themselves symmetric homogeneous polynomials in the generators $\hat{J}^{(j)}_r$. The operator $\underline{\hat{A}}$ can be written as:

$$\underline{\hat{A}} = \sum_j\sum_{N=0}^{N(j)}\sum_{r_1\dots,r_N} a_{r_1,\dots,r_N}\left(g;\hat{C}(j)\right)\{\hat{J}^{(j)}_{r_1}\hat{J}^{(j)}_{r_2}\dots\hat{J}^{(j)}_{r_N}\}_S \tag{2.113}$$

This operator can be mapped into a function:

$$a\left(g,\vec{J}\right) = \sum_j\sum_{N=0}^{N(j)}\sum_{r_1\dots,r_N} a_{r_1,\dots,r_N}(g;C)\; J_{r_1}J_{r_2}\dots J_{r_N} \tag{2.114}$$

In this expression $\vec{J}$ is a collection of J_r, which are the commuting classical variables associated to the canonical momentum coordinates of the classical phase space T^*G, while C are invariant Casimir homogeneous polynomials in them. So, there is a correspondence:

$$\hat{A}\in Op(\mathcal{H}) \iff \underline{\hat{A}}(g)\in b.d.Op(\mathcal{H}_0) \longleftrightarrow a\left(g,\vec{J}\right)\in\mathcal{F}(T^*G)$$

(here *b.d.* means block-diagonal operators) Weyl-Wigner isomorphism sketched along this chapter maps an operator on the whole Hilbert space $\mathcal{H}$ of square integrable functions on the group G to a symbol on what has been called "quantum cotangent space". In this section there has been shown how such a symbol can be mapped into a block diagonal operator on a simpler Hilbert space $\mathcal{H}_0$, where the left action of G is reducible without any degeneracy, and then how this operator can be mapped into a function on the classical cotangent space for the Lie group G.

This chain of invertible maps can be seen as a Weyl-Wigner isomorphism between the space of operators on a Hilbert space, and the set of functions on the cotangent bundle of a compact simple Lie group. This isomorphism then enables to define a noncommutative product in the space of these functions, so to open the possibility to study a new class of noncommutative spaces.

Chapter 3

A fuzzy disc

It is well known that, in the conventional formulation of quantum field theory as the theory of formally quantized classical fields on a classical Minkowski spacetime, ultraviolet divergences arise when one attempts to measure the amplitude of field oscillations at a precise given point in spacetime. These divergences seem to be related to a quantization procedure based on a continuum manifold structure for spacetime.

An analysis of the relations between geometry of spacetime and quantum formalism was started by Dirac [13]. In his effort to describe quantum physics on a classical phase space, he was aware that uncertainty relations led to the impossibility of an infinitely precise localization of points in phase space. This was originated by the noncommutativity among operators representing positions and momenta, whose spectra would classically define the phase space. On a related side, von Neumann was led to study the possibility to replace the continuum phase space structure with a lattice, introducing the idea of smearing out points to Planck cells of area $\sim 2\pi\hbar$. This idea also led to the hypothesis, to cope these ultraviolet divergences, to replace the continuum spacetime with a fundamental lattice. Nevertheless this hypothesis does not fit with the requirement of a natural symmetry action of continuous groups on these approximating spaces.

It was Heisenberg to suggest that one could use a noncommutative structure for spacetime coordinates at very short lenght scale. Noncommutativity would have introduced an ultraviolet cutoff. Snyder [41] was the first to formalize this idea. He wrote that if one assumes that the spectra of spacetime coordinate operators are invariant under Lorentz transformations, then of course the usual spacetime satisfies this requirement. But it is not the only solution: there exists a Lorentz invariant spacetime in which there is a fundamental lenght. This space is related to a set of operators having a Lorentz invariant spectrum. This line was developed by Yang [52], who studied a discrete version of spacetime, on which a larger group (including some sort of translations) properly acts as a symmetry. His work was largely ignored, mostly because, at around the same time, a first renormalization program of quantum field theory finally proved to be successful at accurately predicting numerical values for physical measured quantities in quantum electrodynamics.

Noncommutative geometry [9], considering the topology and the geometry of the space of states as encoded in the algebraic relations among quantum observables, provides a natural formalization for a pointless geometry, and then for quantum field theories [42], and for the analysis of finite approximations to them. Moreover, quantum gravity models, and string theoretical models, suggest the possibility that classical general relativity would break down at a very short lenght scale, spacetime being no longer described by a differentiable manifold [14, 51].

Since in this geometry points are ill-defined, spaces are often thought as fuzzy. The first formalization of the idea of a fuzzy space was introduced by Madore [23], for the sphere[1]. In his approach, a fuzzy sphere is a sequence of nonabelian algebras, more specifically of finite rank matrix algebras, so that at each step of this sequence there is no manifold structure for the set of pure states. Among elements of this sequence, Madore analysed how the fuzzy sphere can be seen as a specific filtration of functions on a sphere. This filtration comes from studying how the sequence of matrix algebras converges towards the set of infinite dimensional diagonal matrices, that is an abelian algebra. This naive notion of convergence was replaced by a meaningful one. M.Rieffel proved that the fuzzy sphere "converges to the sphere" if both each step of the sequence of finite rank matrix algebras and the algebra of functions on the sphere, are seen as compact quantum metric spaces, and the distance among them is the Gromov-Hausdorff distance [37].

A path integral formalism for quantum field theories on these spaces can be introduced. It is based on the substitution of the functional action in an infinite dimensional space with a functional action depending on a finite number of functional degrees of freedom [24]. Quantum fields on these spaces do not present ultraviolet divergences.

The aim of this chapter is to describe a new fuzzy space, the fuzzy disc [22]. It is the first example of a fuzzy approximation of a space with a boundary, and it has been proved to act as a regulator for ultraviolet divergences in the case of a noninteracting field theory.

In the first chapter the Weyl-Wigner approach has been presented as a bridge connecting the quantum to the classical formalism. It can be thought as way to study relations between noncommutative geometry and commutative geometry. Since a fuzzy space can be seen, from the dual point of view of states of those matrix algebras, as a sequence of "nonabelian" spaces, converging to an ordinary, continuum manifold, it is natural to think that Weyl-Wigner formalism can be suited to study these specific models in noncommutative geometry.

In the first section of this chapter the fuzzy sphere is reviewed as the prototype of a fuzzy space. The original approach of Madore is outlined, then it is presented in terms of a Weyl-Wigner isomorphism. This isomorphism is defined used the concept of coherent states for the group $SU(2)$, following Berezin [7]. It is a map from the finite rank matrix algebras to a subset of the space of functions on a 2-sphere: the rank of matrices is related to the dimension of the space on which unitary irreducible representations of the group $SU(2)$ are defined. Noncommutativity in the space of functions on the sphere, via this isomorphism, is related to the rank N of the range matrices, disappearing, as required, in the limit of $N \to \infty$.

Moreover, this isomorphism is explicitly written in terms of the properties

[1] There has also been studied the possibility of a similar approximation for the torus, based on the noncommutative torus algebra [36, 21], and for complex projective spaces [4]

of fuzzy harmonics, that arise in the study of the spectral properties of a fuzzy Laplacian operator, defined in such a way to converge, in the commutative limit, to the ordinary Laplacian operator on the algebra of functions on a sphere.

The end of the first section describes the setting introduced by M.Rieffel to prove the convergence of the fuzzy sphere algebras to the algebra of functions on the sphere.

The second part is devoted to the description of the fuzzy disc. A Weyl-Wigner isomorphism is introduced in terms of functions on a plane, where non-commutativity is represented by a parameter θ. In this formalization, there is no natural concept of a sequence of finite dimensional Hilbert spaces, or finite rank matrix algebras. So it is necessary to introduce a truncation in the algebra of operators, with respect to a specific basis in the Hilbert space. If the dimension of truncation N is constrained to the noncommutativity parameter $N\theta = R^2$, then one obtains a sequence of finite rank matrices, converging towards an abelian algebra of operators, that approximates functions whose support is concentrated on a disc of radius R. On this sequence of states, fuzzy derivatives and a fuzzy Laplacian can be defined, and a system of fuzzy Bessels can be introduced. This set of fuzzy Bessels will be used to define, as fuzzy harmonics in the case of the sphere, a Weyl-Wigner isomorphism between the set of finite rank matrices and the set of functions on a disc. This is the way it is obtained a sequence of finite rank matrix algebras, that converges, in a formal commutative limit, to the algebra of functions on a disc. Moreover, the last section shows how this approximation enables to study a first field theory model, namely a noninteracting one, and how this method works as an ultraviolet regulator.

3.1 The fuzzy sphere as a prototype of a fuzzy space

This section starts with the description of the fuzzy sphere, following the original paper [23]. The aim is to give a first idea of what a fuzzy sphere is, of what is the difference between a fuzzy approximation and the lattice approximation, and of what is a notion, though almost naive, of a limit of the fuzzy sphere to the algebra of functions on the sphere. In the second part of the section the fuzzy sphere will be described making use of a Weyl-Wigner isomorphism, while the third part will explain the exact meaning of that convergence.

The approach that J.Madore used in the introduction of the fuzzy sphere starts from the analysis of the algebra of functions on the sphere S^2. This algebra $\mathcal{C}\left(S^2\right)$ is made of continuous functions on the sphere, and can be seen as the quotient $\mathcal{C}\left(\mathbb{R}^3\right)/\mathcal{I}$, where $\mathcal{I}$ is the two-sided ideal of continuous functions on $\mathbb{R}^3$ whose value is zero on points whose coordinates satisfy[2]:

$$\delta_{ab}x^a x^b = 1 \tag{3.1}$$

[2]The radius of the sphere imbedded in $\mathbb{R}^3$ has been fixed equal to 1.

Functions in $\mathcal{C}\left(\mathbb{R}^3\right)$ have a formal polynomial expansion:

$$\begin{aligned} f(x) &= f_0 + \sum_{a=1}^{3} f_a x^a + \frac{1}{2}\sum_{a,b=1}^{3} f_{ab} x^a x^b + \ldots \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{a_1 \ldots a_l = 1}^{3} x^{a_1} \cdots x^{a_l} \end{aligned} \tag{3.2}$$

Quotienting $\mathcal{C}\left(\mathbb{R}^3\right)$ by the above ideal translates into a set of constraints for the coefficients $f_{a_1 \ldots a_l}$:

- $f_{a_1 \ldots a_l}$ should be totally symmetric in the exchange of indices $a_1 \ldots a_l$ (this requirement actually comes from commutativity of both algebras of functions);
- $f_{a_1 \ldots a_l}$ should be traceless with respect to every pair of indices.

At each order of expansion, represented by l, the number of independent coefficients is $2l+1$, and one has that:

$$\sum_{l=0}^{N} (2l+1) = (N+1)^2$$

This relation is important in this context, because one can consider a truncated expansion of elements of this algebra:

$$f^{(N)}(x) \equiv \sum_{l=0}^{N-1} \frac{1}{l!} \sum_{a_1 \ldots a_l = 1}^{3} f_{a_1 \ldots a_l} x^{a_1} \cdots x^{a_l} \tag{3.3}$$

The number of independent coefficients in $f^{(N)}$ is N^2. This set of functions is no longer an algebra, if they are multiplied via the standard pointwise commutative product. To introduce an algebraic structure into this vector space one could consider coefficients as elements of $\mathbb{C}^{N^2}$. The easiest choice would be to define a commutative componentwise product. Via this definition, the space becomes algebraically isomorphic to the abelian algebra of functions defined on N^2 points. This approximation is that of a lattice.

Nevertheless $\mathbb{C}^{N^2}$ can be seen as the space of $N \times N$ matrices with complex coefficients. Then one can map commutative coordinates into noncommutative coordinates, which are the operators representing the Lie algebra of the group $SU(2)$ on each space $\mathbb{C}^N$:

$$\left[\hat{L}_a^{(N)}, \hat{L}_b^{(N)}\right] = i\epsilon_{abc} \hat{L}_c^{(N)} \tag{3.4}$$

$$x^a \rightarrow \tilde{k} \hat{L}_a^{(N)} \equiv \hat{x}_a^{(N)} \tag{3.5}$$

The space of truncated functions is mapped into the space $\mathbb{M}_N$:

$$f^{(N)} \rightarrow \hat{f}^{(N)} = \sum_{l=0}^{N-1} \frac{1}{l!} \sum_{a_1 \ldots a_l = 1}^{3} f_{a_1 \ldots a_l} \hat{x}_{a_1}^{(N)} \cdots \hat{x}_{a_l}^{(N)} \tag{3.6}$$

This map is well defined because the quotienting relation (3.1) is verified as a Casimir relation for the group $SU(2)$. Fixing the Casimir eigenvalue as the radius of the sphere:

$$\left[\hat{x}_1^{(N)}\right]^2 + \left[\hat{x}_2^{(N)}\right]^2 + \left[\hat{x}_3^{(N)}\right]^2 = 1 \tag{3.7}$$

fixes the value of the constant $\tilde{k}$. This means that noncommuting coordinates satisfy a relation of the form:

$$\left[\hat{x}_a^{(N)}, \hat{x}_b^{(N)}\right] = \frac{2i\epsilon_{abc}}{\sqrt{N^2-1}} \hat{x}_c^{(N)} \tag{3.8}$$

This commutation relation says that, in the formal limit $N \to 0$, generators of this algebra are seen to commute. Moreover, on each truncated subalgebra, isomorphic to the algebra $\mathbb{M}_N$, there is a natural action of the group $SU(2)$, that is the symmetry group acting on the manifold S^2: this would have been impossible in the lattice approximation.

The map (3.6) can be inverted[3]. In the notation of the original paper, this inverse is represented by ϕ_N, and its range is the set of truncated (3.3) functions $f^{(N)}(x)$, which can be seen as symbols of the matrices $\hat{f}^{(N)}$. To consider the limit of this sequence of algebras for $N \to \infty$, Madore stressed that the map ϕ_N is not an algebra morphism, because the algebra of matrices is non abelian: $\phi_N\left(\hat{f}^{(N)}\hat{g}^{(N)}\right)$ is the symbol of the matrix product $\hat{f}^{(N)}\hat{g}^{(N)}$, while $\phi_N\left(\hat{f}^{(N)}\right)\phi_N\left(\hat{g}^{(N)}\right)$ is the pointwise abelian product among symbols $f^{(N)}(x)$ and $g^{(N)}(x)$. The difference:

$$\phi_N\left(\hat{f}^{(N)}\hat{g}^{(N)}\right) - \phi_N\left(\hat{f}^{(N)}\right)\phi_N\left(\hat{g}^{(N)}\right) = o(l/N) \tag{3.9}$$

explicitly shows that the nonabelian product among matrices can be written as a nonabelian product among truncated functions of order N, and that non-commutativity can be estimated to be an infinitesimal term of the order l/N, where l is the degree of the polynomials representing $f^{(N)}(x)$ and $g^{(N)}(x)$. In particular, it can be seen that as the order of l approaches N, the error involved in considering ϕ_N a morphism becomes more and more important.

In the space of matrices a norm is introduced:

$$\parallel \hat{f}^{(N)} \parallel^2 \equiv \frac{1}{N} Tr\left[\hat{f}^{(N)\dagger}\hat{f}^{(N)}\right] \tag{3.10}$$

This norm can be seen as the integral norm on truncated functions on the sphere:

$$\parallel \hat{f}^{(N)} \parallel^2 = \frac{1}{4\pi}\int_{S^2} d\Omega \left|f^{(N)}\right|^2 \tag{3.11}$$

It can be formally checked that:

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} Tr\left[\hat{f}^{(N)}\right] &= \frac{1}{4\pi}\int_{S^2} d\Omega\, f \quad \to \\ \lim_{N\to\infty} \parallel f^{(N)} \parallel &= \parallel f \parallel_{S^2} \end{aligned} \tag{3.12}$$

[3]The invertibility of this map can be proved using the fact that generators $\hat{L}_a^{(N)}$ defines an irreducible representation of the Lie algebra of $SU(2)$. This property has also been used in (2.111). Invertibility of this map will be clarified even in the following subsection.

One can estimate, in this approach, that a generic element $\hat{f}^{(N)} \in \mathbb{M}_N$ (whose elements do not depend on N) has a norm satisfying:

$$\lim_{N\to\infty} \parallel f^{(N)} \parallel = \lim_{N\to\infty} \sqrt{N} \tag{3.13}$$

while a diagonal matrix will have a norm converging to $\parallel f^{(N)} \parallel \to o(1)$ for increasing N. This definition of norm forces to consider only those elements whose limit are diagonal matrices. This choice is the realization of the specific filtration mentioned in the introduction. In this perspective one can naively say that the limit of the fuzzy sphere is a commutative algebra, that "looks like" the algebra of functions on a sphere.

3.1.1 The fuzzy sphere in the Weyl-Wigner formalism

In the previous pages, it has been described how the fuzzy sphere can be looked at as a peculiar sequence of finite rank matrix algebras. It has been stressed how the truncation of the algebra of functions on the sphere can be cast in a matrix form using the properties of the generators of the Lie algebra of $SU(2)$. The main tool to transform a set of functions into a noncommutative algebra is the mapping (3.6), while to study the behaviour of that sequence in the large N limit the main role has been played by its inverse ϕ_N. These can be properly formalised as a Weyl-Wigner isomorphism [15].

The first step of this analysis is to set up an isomorphism between a space of operators and a space of functions. Since a sphere is the coadjoint orbit of the group $SU(2)$, the basic tool to introduce this map is a system of coherent states, specialising the general arguments of appendix A.3.

To define a system of coherent states the first problem is the study of unitary irreducible representations for the group. It is well known which are these UIRR's. On each finite dimensional Hilbert space $\mathbb{C}^N$, with $N = 2L + 1$ $(L = 0, 1/2, 1, 3/2 \ldots)$, where a basis is given by vectors that, in ket notation, are represented as $\mid L, M\rangle$ with $M = (-L, -L+1 \ldots L-1, L)$, one has:

$$u \in SU(2) \overset{\hat{R}^{(L)}}{\mapsto} \mathcal{B}\left(\mathbb{C}^N\right) \tag{3.14}$$

The matrix elements of this representation are given by:

$$\langle L, M \mid \hat{R}^{(L)}(u) \mid L, M'\rangle = D^L_{MM'}(u) \tag{3.15}$$

These[4] are called Wigner functions [45].

The second step is to fix a fiducial state. One can choose the so called highest weight in the representation: $\mid \psi_0\rangle = \mid L, L\rangle$. If the group manifold is parametrised by Euler angles, then u represents a point whose "coordinates" range through $\alpha \in [0, 4\pi)$, $\beta \in [0, \pi)$, $\gamma \in [0, 2\pi)$. Fixed the fiducial vector, its stability subgroup H_{ψ_0} by the $\hat{R}^{(L)}$ representation is made by elements for which $\beta = 0$ (this condition can be seen to be valid whatever the dimension N of the space of representation is).

Two elements u and u' are equivalent if $u^\dagger u' \in H_{\psi_0}$. It is possible to prove that

$$SU(2)/H_{\psi_0} \approx S^2 \tag{3.16}$$

[4] These Wigner functions are a specific example of the general definition used in chapter 2, for matrix elements of UIRR of a Lie group (2.50).

identifying $\theta = \beta$ and $\varphi = \alpha / mod\, 2\pi$. Chosen a representative $\tilde{u}$ element in each equivalence class of the quotient, the set of coherent states is defined as:

$$\mid \theta, \varphi, N \rangle = \hat{R}^{(L)}\left(\tilde{u}\right) \mid L, L \rangle \tag{3.17}$$

The left hand side ket now explicitly depends on N, the dimension of the space on which the representation takes place. Projected on the basis elements, one has:

$$\begin{aligned}
\langle L, M \mid \theta, \varphi, N \rangle &= D^{L}_{ML}\left(\tilde{u}\right) \\
\mid \theta, \varphi, N \rangle &= \sum_{M=-L}^{L} \left[\frac{(2L)!}{(L+M)!\,(L-M)!} \right]^{1/2} \left(\cos\theta/2\right)^{L+M} \left(\sin\theta/2\right)^{L-M} e^{-i\varphi M} \mid L, M \rangle
\end{aligned} \tag{3.18}$$

This set of states is nonorthogonal, and overcomplete:

$$\begin{aligned}
\langle \theta', \varphi', N \mid \theta, \varphi, N \rangle &= e^{-iL\left(\varphi' - \varphi\right)} \left[e^{i\left(\varphi' - \varphi\right)} \cos\theta/2 \cos\theta'/2 + \sin\theta/2 \sin\theta'/2 \right]^{2L} \\
\mathbf{1} &= \frac{2L+1}{4\pi} \int_{S^2} d\Omega \mid \theta, \varphi, N \rangle \langle \theta, \varphi, N \mid
\end{aligned} \tag{3.19}$$

Using this set of vectors it is possible to define a map from the space of operators on a finite dimensional Hilbert space to the space of functions on the sphere S^2:

$$\begin{aligned}
\hat{A}^{(N)} \in \mathcal{B}\left(\mathbb{C}^N\right) \approx \mathbb{M}_N \quad &\mapsto \quad A^{(N)} \in \mathcal{F}\left(S^2\right) \\
A^{(N)}\left(\theta, \varphi\right) \quad &= \quad \langle \theta, \varphi, N \mid \hat{A}^{(N)} \mid \theta, \varphi, N \rangle
\end{aligned} \tag{3.20}$$

So this is a way to map every finite rank matrix into a function on a sphere, called Berezin symbol. Among these operators, there are $\hat{Y}^{(N)}_{JM}$ whose symbols are the spherical harmonics, up to order $2L$ (here $J = 0, 1, \ldots, 2L$ and $M = -J, \ldots, +J$):

$$\langle \theta, \varphi, N \mid \hat{Y}^{(N)}_{JM} \mid \theta, \varphi, N \rangle \; = \; Y_{JM}\left(\theta, \varphi\right) \tag{3.21}$$

These operators are called *fuzzy harmonics*. The origin of this name must be traced back to the definition, on each finite rank matrix algebra $\mathbb{M}_N$, of an operator, in terms of the generators (3.4) $\hat{L}^{(N)}_a$ representing the Lie algebra of the group $SU\left(2\right)$ on the space $\mathbb{C}^N$:

$$\begin{aligned}
\nabla^2 \quad &: \quad \mathbb{M}_N \mapsto \mathbb{M}_N \\
\nabla^2 \hat{A}^{(N)} \quad &= \quad \left[\hat{L}^{(N)}_s, \left[\hat{L}^{(N)}_s, \hat{A}^{(N)} \right] \right]
\end{aligned} \tag{3.22}$$

This operator is called *fuzzy Laplacian.* It can be seen that the spectrum of this fuzzy Laplacian is given by eigenvalues $\mathcal{L}_j = j\left(j+1\right)$, where $j = 0, \ldots, 2L$, and every eigenvalue has a multiplicity of $2j+1$. The spectrum of this fuzzy Laplacian thus coincides with the spectrum of the continuum Laplacian defined in the space of functions on a sphere, up to order $2L$. The cut-off of this spectrum is of course related to the dimension of the rank of the matrix algebra under analysis. Fuzzy harmonics are the eigenstates of this operator, or so to say, "eigenmatrices" of this fuzzy Laplacian.

Fuzzy harmonics are a basis in each space of matrices $\mathbb{M}_N$. They are trace orthogonal with respect to the scalar product:

$$Tr\left[\left(\hat{Y}_{JM}^{(N)}\right)^{\dagger}\hat{Y}_{J'M'}^{(N)}\right] = \lambda_{NJM}\,\delta_{JJ'}\delta_{MM'} \tag{3.23}$$

In the space of finite rank matrices of order $N = 2L+1$, that is the set of operators $\mathbb{M}_N$, it is possible to introduce a set of N^2 polarization operators [45] $\hat{T}_{JM}^{(N)}$. They satisfy:

$$\begin{aligned}\left[\hat{L}_{\mu}^{(N)},\hat{T}_{JM}^{(N)}\right] &= \sqrt{L\,(L+1)}\,C_{LM\,1\mu}^{L\,M+\mu}\,\hat{T}_{LM+\mu}^{(N)}\\ Tr\left[\hat{T}_{JM}^{(N)\,\dagger}\,\hat{T}_{J'M'}^{(N)}\right] &= \delta_{JJ'}\delta_{MM'}\\ \hat{T}_{JM}^{(N)\,\dagger} &= (-1)^{M}\,\hat{T}_{J-M}^{(N)}\end{aligned} \tag{3.24}$$

These three conditions completely determine the polarization operators: fuzzy harmonics are proportional to polarization operators:

$$\hat{Y}_{JM}^{(N)} = \sqrt{\lambda_{NJM}}\,\hat{T}_{JM}^{(N)} \tag{3.25}$$

Using this basis, an element $\hat{F}^{(N)}$ belonging to $\mathbb{M}_N$ can be expanded as:

$$\hat{F}^{(N)} = \sum_{J=0}^{2L}\sum_{M=-J}^{J} F_{JM}^{(N)}\,\hat{Y}_{JM}^{(N)} \tag{3.26}$$

Coefficients of this expansion are given by:

$$F_{JM}^{(N)} = Tr\left[\hat{Y}_{JM}^{(N)\dagger}\,\hat{F}^{(N)}\right]/\lambda_{NJM} \tag{3.27}$$

A Weyl-Wigner map can be defined simply mapping spherical harmonics into fuzzy harmonics:

$$\hat{Y}_{JM}^{(N)} \Leftrightarrow Y_{JM}(\theta,\varphi) \tag{3.28}$$

This map clearly depends on the dimension N of the space on which fuzzy harmonics are realized. It can be linearly extended by:

$$\hat{F}^{(N)} = \sum_{J=0}^{2L}\sum_{M=-J}^{J} F_{JM}^{(N)}\hat{Y}_{JM}^{(N)} \;\leftrightarrow\; F^{(N)}(\theta,\phi) = \sum_{J=0}^{2L}\sum_{M=-J}^{+J} F_{JM}^{(N)}Y_{JM}(\theta,\phi) \tag{3.29}$$

This is a Weyl-Wigner isomorphism. How can this formalization be used to define a fuzzy sphere? Given a function on a sphere, if it is square integrable with respect to the standard measure $d\Omega = d\varphi\,\sin\theta d\theta$, then it can be expanded in the basis of spherical harmonics:

$$f(\theta,\varphi) = \sum_{J=0}^{\infty}\sum_{M=-J}^{J} f_{JM}\,Y_{JM}(\theta,\varphi) \tag{3.30}$$

This expansion can be truncated:

$$f^{(N)}(\theta,\varphi) = \sum_{J=0}^{2L}\sum_{M=-J}^{J} f_{JM}\,Y_{JM}(\theta,\varphi) \tag{3.31}$$

This is a set of functions whose expansion in spherical harmonics is up to order $2L = N-1$. It is a vector space, but it is no more an algebra, with the standard definition of sum and pointwise product of two functions, as the product of two spherical harmonics of order say $2L$ has spherical components of order larger[5] than $2L$. If these truncated functions are mapped, via the Weyl-Wigner procedure, into matrices:

$$f^{(N)}(\theta,\varphi) \mapsto \hat{f}^{(N)} = \sum_{J=0}^{2L} \sum_{M=-J}^{+J} f_{JM} \hat{Y}_{JM}^{(N)} \tag{3.33}$$

then this set of truncated functions is given the vector space structure of $\mathbb{M}_N$. Nevertheless $\mathbb{M}_N$ is even an algebra, a non abelian algebra. Invertibility of this association (3.33) enables to define, in the set of the truncated functions on the sphere, a non abelian product, isomorphic to that of matrices[6]:

$$\left(f^{(N)} * g^{(N)}\right)(\theta,\varphi) = \sum_{J=0}^{2L} \sum_{M=-J}^{J} Tr\left[\hat{f}^{(N)} \hat{g}^{(N)} \hat{Y}_{JM}^{(N)\dagger}\right] Y_{JM}(\theta,\varphi) / \lambda_{NJM} \tag{3.35}$$

The Weyl-Wigner map (3.29) has been used to make each set of truncated functions a non abelian algebra $\mathcal{A}^{(N)}\left(S^2, *\right)$, isomorphic to $\mathbb{M}_N$. These algebras can be seen as formally generated by matrices which are the images of the norm 1 vectors in $\mathbb{R}^3$, that are points on a sphere. They are mapped into multiples of the generators $\hat{L}_a^{(N)}$ of the Lie algebra:

$$\frac{x_a}{\| \vec{x} \|} \mapsto \hat{x}_a^{(N)} \qquad \left[\hat{x}_a^{(N)}, \hat{x}_b^{(N)}\right] = \frac{2i\varepsilon_{abc}}{\sqrt{N^2-1}} \hat{x}_c^{(N)} \tag{3.36}$$

This relation perfectly fits with (3.8): once more, the commutation rules satisfied by generators of the algebras in the sequence $\mathcal{A}^{(N)}\left(S^2, *\right)$ make it intuitively clear that the limit for $N \to \infty$ of this sequence is an abelian algebra. This is the reason why this sequence is called *fuzzy sphere*.

The formal proof of this convergence towards the algebra of functions on a sphere has been given by Rieffel, in terms of the so called Quantum Gromov-Hausdorff distance among quantum metric spaces.

3.1.2 An analysis of the convergence of matrix algebras to the sphere

In a series of papers, M.Rieffel studied the problem of giving a precise meaning to the notion of the convergence of the fuzzy sphere to the classical sphere, that

[5] The product of two spherical harmonics is:

$$\left(Y_{J'M'} Y_{J''M''}\right)(\theta,\varphi) = \sum_{J=|J'-J''|}^{J=|J'+J''|} \sum_{M=-J}^{J} \sqrt{\frac{(2J'+1)(2J''+1)}{4\pi(2J+1)}} C_{J'0\,J''0}^{J0} C_{J'M'\,J''M''}^{JM} Y_{JM}(\theta,\varphi) \tag{3.32}$$

in terms of Clebsh-Gordan coefficients for $SU(2)$.

[6] The product of two fuzzy harmonics can be obtained by the product of two polarization operators [45]:

$$\hat{T}_{J'M'}^{(N)} \hat{T}_{J''M''}^{(N)} = \sum_J (-1)^{2L+J} \sqrt{(2J'+1)(2J''+1)} \left\{ \begin{matrix} J' & J'' & J \\ L & L & L \end{matrix} \right\} C_{J'M'\,J''M''}^{JM} \hat{T}_{JM}^{(N)} \tag{3.34}$$

is the convergence of that sequence of finite rank matrix algebras to the algebra of functions on a sphere. The aim of this section is to report some aspects of this analysis. Specifically, it will be sketched how these algebras can be considered as elements of a peculiar metric space, where the convergence to the algebra of functions on the sphere can be formalized.

In noncommutative geometry, the natural way to specify a metric is by means of a suitable "Lipschitz seminorm". This idea was developed by Connes [10]. He pointed out that from a Lipschitz seminorm one obtains in a simple way an ordinary metric on the state space of a C^*-algebra.

Given a compact metric space (Z, ρ) (ρ is an ordinary metric), a Lipschitz seminorm is defined for functions on Z (x, y are points of Z):

$$L_\rho(f) \equiv \sup\{|f(x) - f(y)| / \rho(x, y) : x \neq y\} \tag{3.37}$$

This is actually a seminorm as it is 0 for constant functions, and can take the value $+\infty$, unless the domain is restricted to Lipschitz functions. From L_ρ the metric ρ can be obtained:

$$\rho(x, y) = \sup\{|f(x) - f(y)| : L_\rho(f) \leq 1\} \tag{3.38}$$

This metric on pure states can be extended to the set of all states (probability measures on Z):

$$\rho(\mu, \nu) \equiv \sup\{|\mu(f) - \nu(f)| : L_\rho(f) \leq 1\} \tag{3.39}$$

This metric induces, in the set of states $\mathcal{S}(Z)$ of the algebra of functions on Z, a topology that coincides with the weak $*$-topology that $\mathcal{S}(Z)$ would have if it were considered as the dual of the C^*-algebra $C(Z)$ of continuous functions on Z[7].

These results are used to extend the notion of metric to states on noncommutative algebras. If $\mathcal{A}$ is a unital C^*-algebra, and L is a Lipschitz seminorm on this algebra, satisfying $L(a) = L\left(a^\dagger\right)$, then, on the set of states $\mathcal{S}(\mathcal{A})$, a metric is defined by (a is an element of the algebra, μ, ν are states of the algebra):

$$\rho(\mu, \nu) \equiv \sup\{|\mu(a) - \nu(a)| : L_\rho(a) \leq 1\} \tag{3.40}$$

If this metric induces on $\mathcal{S}(\mathcal{A})$ the weak $*$-topology, then $(\mathcal{A}, L)$ is a *compact quantum metric space*. The word "quantum" is used to stress that its origin lies in noncommutative geometry.

The notion of Gromov-Hausdorff distance for compact quantum metric spaces is an evolution of the "classical" Hausdorff distance for compact metric spaces.

If (Z, ρ) is again a compact metric space, and X is a closed subset $X \subset Z$, an r-neighborood of X of radius r is given by all the points of Z whose "distance" from X is less than r:

$$\mathcal{N}_r^\rho(X) \equiv \{z \in Z : \exists x \in X : \rho(x, z) < r\} \tag{3.41}$$

If $X \subset Z$ and $Y \subset Z$ are two such subsets, the Hausdorff distance between them is:

$$d_{\mathcal{H}}^\rho(X, Y) \equiv \inf\{r : X \subseteq \mathcal{N}_r^\rho(Y) ; Y \subseteq \mathcal{N}_r^\rho(X)\} \tag{3.42}$$

[7] If $\mathcal{A}^*$ is the space of states of a suitable algebra $\mathcal{A}$, i.e. the Banach space of norm 1 continuous linear functionals $\phi : \mathcal{A} \mapsto \mathbb{C}$ then the weak $*$-topology is given as the pointwise convergence on elements on $\mathcal{A}$.

Now it is possible to consider (X, ρ_X) and (Y, ρ_Y), a pair of independent compact metric spaces. $\mathcal{U} \equiv X \underline{\cup} Y$ is the disjoint union of the two. $\mathcal{M}(\rho_X, \rho_Y)$ is the set of metrics on $\mathcal{U}$ such that ρ_X is the quotient on X and ρ_Y is the quotient on Y. The Gromov-Hausdorff distance between X and Y is:

$$d_{\mathcal{GH}}(X, Y) \equiv \inf\{d^{\rho}_{\mathcal{H}}(X, Y) : \rho \in \mathcal{M}(\rho_X, \rho_Y)\} \tag{3.43}$$

This reasoning is extended to compact quantum metric spaces.

Given a pair of compact quantum metric spaces $(\mathbb{A}, L_{\mathbb{A}})$ and $(\mathbb{B}, L_{\mathbb{B}})$ (here $\mathbb{A}$ and $\mathbb{B}$ can be considered as operator algebras, while $L_{\mathbb{A}}$ and $L_{\mathbb{B}}$ are Lipschitz seminorm), one can consider the "disjoint" union:

$$(\mathbb{A} \oplus \mathbb{B}, \mathcal{M}(L_{\mathbb{A}}, L_{\mathbb{B}}))$$

where $\mathcal{M}(L_{\mathbb{A}}, L_{\mathbb{B}})$ is the set of Lipschitz seminorm whose quotient is $L_{\mathbb{A}}$ on $\mathbb{A}$ and $L_{\mathbb{B}}$ on $\mathbb{B}$. On the set of states $\mathcal{S}(\mathbb{A} \oplus \mathbb{B})$ one can induce a metric ρ_L from a Lipschitz seminorm L in $\mathcal{M}(L_{\mathbb{A}}, L_{\mathbb{B}})$ via (3.40), and then has an Hausdorff distance 3.42 between the states of $\mathbb{A}$, $\mathcal{S}(\mathbb{A})$, and the states of $\mathbb{B}$, $\mathcal{S}(\mathbb{B})$, seen as subsets of $\mathcal{S}(\mathbb{A} \oplus \mathbb{B})$. The Gromov-Hausdorff distance between $\mathbb{A}$ and $\mathbb{B}$ is:

$$d_{\mathcal{GH}}(\mathbb{A}, \mathbb{B}) \equiv \inf\{d^{\rho_L}_{\mathcal{H}}(\mathcal{S}(\mathbb{A}), \mathcal{S}(\mathbb{B})) : L \in \mathcal{M}(L_{\mathbb{A}}, L_{\mathbb{B}})\} \tag{3.44}$$

The notion of Gromov-Hausdorff distance is then well adapted to study problems of convergence is a space of suitable algebras. The next step is to express the algebra of functions on a sphere in this context.

Let G be a compact Lie group, and $U(g)$ a unitary irreducible representation on a finite dimensional Hilbert space $\mathbb{C}^N$. The set $\mathcal{B}\left(\mathbb{C}^N\right) = \mathcal{B}^{(N)} = \mathbb{M}_N$ of bounded operators on this space, which are finite rank matrices, is a unital C^*-algebra. The group G acts on this space of operators. This action α is a conjugation, and maps the space $\mathcal{B}\left(\mathbb{C}^N\right)$ into itself:

$$\alpha(g) \cdot T \equiv U(g) \cdot T \cdot U\left(g^{-1}\right) \tag{3.45}$$

This action is "ergodic", in the sense that the only invariant element is a multiple of the identity $T = c\mathbf{1}$, because the representation U is irreducible.

It can be introduced a function on elements of G, measuring a "lenght" from the identity:

- $l(g) \geq 0$. Moreover, $l(g) = 0$ iff $g = e$
- $l(g) = l\left(g^{-1}\right)$
- $l(g g') \leq l(g) + l(g')$
- $l(g) = l\left(g' g g'^{-1}\right)$

On the space of matrices $\mathcal{B}^{(N)}$ a Lipschitz seminorm can be defined:

$$L_{\mathcal{B}}(T) \equiv \sup\{\| \alpha(g) \cdot T - T \| / l(g) : l(g) \neq 0\} \tag{3.46}$$

The space $\left(\mathcal{B}\left(\mathbb{C}^N\right), L_{\mathcal{B}}\right)$ is a compact quantum metric space.

In the set of operators $\mathcal{B}\left(\mathbb{C}^N\right)$ one can choose a rank one projector P, and use it to define the Berezin covariant symbol of an operator T:

$$\begin{aligned}\sigma_T(g) &= Tr\left[T\left(\alpha(g)\cdot P\right)\right] \\ &= Tr\left[TU(g)\,PU^{\dagger}(g)\right]\end{aligned} \tag{3.47}$$

It is evident that, if H is the isotropy subgroup of P for the action of α, then the symbol $\sigma_T(g)$ is clearly a function on the quotient space G/H, whose points are represented by x. The algebra of functions on this quotient is $\mathcal{A}(G/H)$. This symbol is related to the symbol introduced in (3.20). If P is considered in the form $\mid\psi_0\rangle\langle\psi_0\mid$, then $U(g)\mid\psi_0\rangle$ is a system of coherent states for the group G, and the ciclicity of the trace proves that the symbol $\sigma_T(x)$ can be written as the mean value of T on the coherent states labelled by x. The identification of this symbol with the one defined in (3.20) for the sphere is obtained by the requirement that this symbol were zero only in correspondence to the matrix $T=0$ (usually this condition is referred to as a faithfulness condition on the mapping σ_T). This happens only if the compact group G is semisimple, and the projector P ranges over the highest weight vector in the Cartan analysis[8].

There is a natural left action of the group G on this algebra:

$$(\lambda_g f)(x) \equiv f\left(g^{-1}\cdot x\right) \tag{3.48}$$

where $g\cdot x$ represents the left multiplication action of the element $g\in G$ on the equivalence class in G/H whose label is x. Also the lenght function can be quotiented to a lenght function $\tilde{l}$ on G/H[9]. Equipped with the Lipschitz seminorm:

$$L_{\mathcal{A}}(f) \equiv \sup\left\{\left|f(x)-(\lambda_g f)(x)\right|/\tilde{l}(x)\,:\,x\neq e\right\} \tag{3.49}$$

one has that $(\mathcal{A},L_{\mathcal{A}})$ is a compact quantum metric space.

The fuzzy sphere is a sequence of finite rank matrix algebras, obtained as the set of operators on *each* finite dimensional Hilbert space $\mathbb{C}^N$, because a unitary irreducible representation of the group $SU(2)$ is allowed on such space for each N. Moreover, the symbol (3.20) is introduced using a set of coherent states whose defining fiducial projector ranges over the highest weight vector. This means that the fuzzy sphere can be seen as a sequence of compact quantum metric spaces, with Lipschitz seminorm $L_{\mathcal{B}^{(N)}}$.

It is now possible to consider the "distance" between $\mathcal{B}^{(N)}$ and $\mathcal{A}$ in the space $\left(\mathcal{B}^{(N)}\oplus\mathcal{A},\mathcal{M}\left(L_{\mathcal{B}^{(N)}},L_{\mathcal{A}}\right)\right)$. The main result is that the sequence of Gromov-Hausdorff distances:

$$\lim_{N\to\infty} d^{(N)}_{\mathcal{GH}}\left(\mathcal{B}^{(N)},\mathcal{A}\right) = 0 \tag{3.50}$$

This is the meaning of the sentence that the fuzzy sphere converges to the algebra of functions on a sphere.

[8] This condition means that, if $\mid\psi_0\rangle$ is the highest weight vector for the considered representation $U(g)$, then $\tilde{U}(E_+)\mid\psi_0\rangle=0$, where $\tilde{U}$ is the representation of the Lie algebra $\underline{G}$ induced by U, and E_+ are the positive Cartan's roots in $\underline{G}$.

[9] In the case of compact semi-simple Lie group an admissible lenght function is given by the geodesics lenght with respect to the Cartan-Killing metric (2.65).

3.2 A fuzzy disc

In the previous section, the fuzzy sphere has been introduced via a kind of Weyl-Wigner formalism, which made great use of the fact that the 2-dimensional sphere is the coadjoint orbit of the group $SU(2)$, that is compact. Compactness of the group brought to a natural definition of a sequence of finite dimensional Hilbert spaces, that is to a natural identification of the dimension of these spaces as a cut-off index for a suitable expansion of a generic function on the sphere. Moreover, compactness of the group played a fundamental role in the analysis of the convergence performed by M.Rieffel. Properties of the sphere as an orbit of that group made it possible to formalize every point of the sphere as a coherent state, defined on every of those finite dimensional space carrying the UIRR's. So the map between operators (finite rank matrices) and functions on the sphere has been introduced via the natural Berezin procedure, and has been refined stressing the role of the fuzzy Laplacian operator in the definition of a fuzzy harmonics system as a basis in the set $\mathbb{M}_N$.

It would be intuitively natural, to draw a path towards the definition of a fuzzy disc, to analyse the possibility that a disc were a coadjoint orbit for a Lie group. If this were the case, it would be possible to introduce a system of coherent states labelled by its points, and some sort of Berezin map [7] between operators and a suitable set of functions on the disc.

This, in some sense naive, approach meets some troubles. There is a group, from which it is possible to define a system of coherent states in correspondence with points of a disc. This group is $SU(1,1)$, but it is non compact, so its UIRR's are not realized on finite dimensional Hilbert spaces. There is no more an intrinsic concept of a cut-off index, the dimension of the fuzzyfication, in this context.

To pursue the task of defining a fuzzy version of the algebra of functions on a disc, it is possible to follow, and extend, the second line of the path sketched in sections above. This will circumvent the problem of an intrinsic definition of fuzzyfication dimension. It is well known that a basis for the space of functions on a disc is related to a suitable system of Bessel functions. As the Weyl-Wigner map for functions on a sphere has been introduced (3.29) mapping spherical harmonics into fuzzy harmonics, is it possible to define a system of *fuzzy Bessels* in terms of finite rank matrices, and mapping again continuum Bessels into them?

Since Bessel functions are defined on a plane, introduced to solve a class of boundary values problems for a Laplacian operator, the first step of this analysis will be the introduction of a specific noncommutative plane. In this algebra it will be then introduced a system of "fuzzy" derivations, and a "fuzzy" Laplacian, mimicing, as much as possible, the properties of the continuum version. The final answer to the problem will be given studying the spectral resolution of this fuzzy Laplacian.

3.2.1 A noncommutative plane as a matrix algebra

In section 1.6 it has been shown how the Weyl map defined in the first chapter can be obtained in terms of a set of quantizer operators arising in the study of a specific unitary representation of the Heisenberg-Weyl-Wigner group. In that section a system of generalised coherent states for this group has been introduced, and it has been seen that such coherent states are in correspondence with points of a complex plane. But they have been used just to define an action of the HWW group on the plane, and to select which elements of the group act as a reflections on the plane.

In this section a noncommutative plane will be defined. A Weyl-Wigner map will be introduced following the general procedure of Berezin: so the first step will be the definition of a set of generalised coherent states for the Heisenberg-Weyl group, since they are labelled by points of a plane.

An analysis of the Heisenberg-Weyl group has already been performed in section 1.6. Since in this chapter the identification of the two dimensional plane with the phase space carrying a classical dynamics of a one dimensional point particle will be definitely abandoned, it will be assumed a system of coordinates of the kind (x, y) for a point of $\mathbb{R}^2$; θ will be a parameter introducing an explicit noncommutativity.

With this notation, Heisenberg-Weyl group is a manifold $\mathbb{R}^3$, whose points are represented by a triple (x, y, λ), with the composition rule:

$$(x, y, \lambda) \cdot (x', y', \lambda') = \left(x + x', y + y', \lambda + \lambda' - \frac{1}{\theta}(xy' - x'y)\right) \tag{3.51}$$

The identity of this group is given by:

$$id_W = (0, 0, 0) \tag{3.52}$$

and the inverse of a generic element is:

$$(x, y, \lambda)^{-1} = (-x, -y, -\lambda) \tag{3.53}$$

Complexifying the plane via $z = x + iy$ and $\overline{z} = x - iy$ the group law acquires the form:

$$(z, \lambda) \cdot (z', \lambda') = \left(z + z', \lambda + \lambda' + \frac{i}{2\theta}(\overline{z}z' - \overline{z}'z)\right) \tag{3.54}$$

The Hilbert space on which representing this group is again the Fock space $\mathcal{F}$ (1.120) of finite norm complex analytical functions in the w variable where the norm is obtained by the scalar product:

$$\langle f \mid g \rangle \equiv \int \frac{d^2 w}{\pi\theta}\, e^{-\bar{w}w/\theta}\, \bar{f}(w)\, g(w) \tag{3.55}$$

and the orthonormal basis chosen is given by:

$$\psi_n(w) = \frac{w^n}{\sqrt{\theta^n n!}} \tag{3.56}$$

The unitary representation of the Heisenberg-Weyl group is given by operators $\hat{T}(z, \lambda) \in Op(\mathcal{F})$:

$$\left(\hat{T}f\right)(w) = e^{i\lambda} e^{-\bar{z}z/2\theta} e^{zw/\theta} f(w - \bar{z}) \tag{3.57}$$

Chosen the first basis element $\psi_0(w)$ as fiducial state, the procedure already outlined gives a set of coherent states perfectly coincident with the system obtained for the more general Heisenberg-Weyl-Wigner group (1.123). In the realization of the Fock space as complex analytical functions, a coherent state is then:

$$\mid z\rangle \;\rightarrow\; \psi_{(z)}(w) = e^{-\bar{z}z/2\theta} e^{zw/\theta} \tag{3.58}$$

and, with ψ_n an element of the basis already considered:

$$| z\rangle = \sum_{n=0}^{\infty} e^{-\bar{z}z/2\theta} \frac{\bar{z}^n}{\sqrt{n!\theta^n}} | \psi_n\rangle \tag{3.59}$$

Such coherent states are non orthogonal, and overcomplete:

$$\begin{aligned} \langle z \mid z'\rangle &= e^{-\left(|z|^2+|z'|^2-2\bar{z}z'\right)/\theta} \\ \mathbf{1} &= \int \frac{d^2 z}{\pi\theta} \mid z\rangle\langle z \mid \end{aligned} \tag{3.60}$$

On this Hilbert space $\mathcal{F}$, it is possible to introduce a pair of creation-annihilation operators:

$$\begin{aligned} (\hat{a}f)(w) &= \theta\frac{df}{dw} \\ \left(\hat{a}^\dagger f\right)(w) &= wf(w) \end{aligned} \tag{3.61}$$

such that

$$\left[\hat{a}, \hat{a}^\dagger\right] = \theta \tag{3.62}$$

In the chosen orthonormal basis, one has:

$$\begin{aligned} \hat{a} \mid \psi_n\rangle &= \sqrt{n\theta} \mid \psi_{n-1}\rangle \\ \hat{a}^\dagger \mid \psi_n\rangle &= \sqrt{(n+1)\theta} \mid \psi_{n+1}\rangle \end{aligned} \tag{3.63}$$

and

$$\begin{aligned} \langle z \mid \hat{a} \mid z\rangle = z \\ \langle z \mid \hat{a}^\dagger \mid z\rangle = \bar{z} \end{aligned} \tag{3.64}$$

These relations can be extended. A Berezin symbol can be associated to an operator in the Fock space:

$$f(\bar{z}, z) = \langle z \mid \hat{f} \mid z\rangle \tag{3.65}$$

This can be seen as a Wigner map. It can be inverted:

$$\hat{f} = \int \frac{d^2\xi}{\pi\theta} \int \frac{d^2 z}{\pi\theta} f(z, \bar{z})\, e^{-\left(\bar{z}\xi-\bar{\xi}z\right)/\theta} e^{\xi\hat{a}^\dagger/\theta} e^{-\bar{\xi}\hat{a}/\theta} \tag{3.66}$$

This quantization map for functions on a plane can be given an interesting form. A first analysis can be restricted to functions which can be written as Taylor series in $\bar{z}, z$:

$$f(\bar{z}, z) = \sum_{m,n=0}^{\infty} f_{mn}^{Tay} \bar{z}^m z^n \tag{3.67}$$

An easy calculation says that this f is the symbol of the operator:

$$\hat{f} = \sum_{m,n=0}^{\infty} f_{mn}^{Tay} \hat{a}^{\dagger m} \hat{a}^n \tag{3.68}$$

The second analysis starts from an operator written in a density matrix notation:

$$\hat{f} = \sum_{m,n=0}^{\infty} f_{mn} \mid \psi_m\rangle\langle\psi_n \mid \tag{3.69}$$

The Berezin symbol of this operator is the function:

$$f(\bar{z}, z) = e^{-|z|^2/\theta} \sum_{m,n=0}^{\infty} f_{mn} \frac{\bar{z}^m z^n}{\sqrt{m!n!\theta^{m+n}}} \tag{3.70}$$

Given the Taylor coefficients f_{mn}^{Tay}, one has:

$$f_{lk} = \sum_{q=0}^{\min(l,k)} f_{l-q\,k-q}^{Tay} \frac{\sqrt{k!l!\theta^{l+k}}}{q!\theta^q} \tag{3.71}$$

while the inverse relation is given by:

$$f_{mn}^{Tay} = \sum_{p=0}^{\min(m,n)} \frac{(-1)^p}{p!\sqrt{(m-p)!\,(n-p)!\theta^{m+n}}} f_{m-p\,n-p} \tag{3.72}$$

Equation (3.68) shows that the quantization of a monomial in the variables $z, \bar{z}$ is an operator in $\hat{a}, \hat{a}^\dagger$, formally a monomial in these two noncommuting variables, with all terms in $\hat{a}^\dagger$ acting at the left side with respect to terms in $\hat{a}$. This means that this quantization brings a specific ordering. In section (1.5.3) it has been analysed how ordering is usually related to a weight in the Weyl map that defines the quantization. The Weyl map (3.66) can be written, restoring real variables, as (here $u = (a + ib)/2$):

$$\hat{f} = \int \frac{dadb}{2\pi\theta} \int \frac{dxdy}{2\pi\theta} f(x, y)\; e^{-i(bx-ay)/\theta}\, e^{u\hat{a}^\dagger/\theta}\, e^{-\bar{u}\hat{a}/\theta} \tag{3.73}$$

This expression is equal to:

$$\hat{f} = \int \frac{dadb}{2\pi\theta} \tilde{f}(b, a)\; e^{\left(a^2+b^2\right)/8\theta}\, \hat{D}\left(a/\sqrt{2}, b/\sqrt{2}\right) \tag{3.74}$$

The presence of the factor $1/\sqrt{2}$ in the argument of the Displacement operator follows from the specific complexification of the plane via $z = x + iy$. It can be compared to equation(1.107). The factor $e^{\left(a^2+b^2\right)/8\theta}$ is a weight to the standard Weyl map, so the quantization via Berezin procedure is just an example of a weighted quantization.

The invertibility of Weyl map (on a suitable domain of functions on the plane) enables to define a noncommutative product in the space of functions. It is known as Voros product:

$$(f * g)(\bar{z}, z) = \langle z \mid \hat{f}\hat{g} \mid z\rangle \tag{3.75}$$

It is a non local product:

$$(f * g)(\bar{z}, z) = e^{-\bar{z}z/\theta} \int \frac{d^2\xi}{\pi\theta} f(\bar{z}, \xi)\, g\left(\bar{\xi}, z\right)\, e^{-\bar{\xi}\xi/\theta} e^{\bar{\xi}z/\theta} e^{\bar{z}\xi/\theta} \tag{3.76}$$

Its asymptotic expansion acquires the form:

$$(f * g)(\bar{z}, z) = f\, e^{\theta \overleftarrow{\partial}_{\bar{z}} \overrightarrow{\partial}_z} g \tag{3.77}$$

and makes it clear that it is a deformation, in θ, of the pointwise commutative product. Since it is the translation, in the space of functions, of the product in the space of operators, given symbols expressed in the form (3.70), the product acquires a matrix form:

$$(f * g)_{mn} = \sum_{k=0}^{\infty} f_{mk} g_{kn} \tag{3.78}$$

The space of functions on the plane, with the standard definition of sum, and the product given by the Voros product (3.75), is a nonabelian algebra, a noncommutative plane. This algebra $\mathcal{A}_\theta = \left(\mathcal{F}\left(\mathbb{R}^2\right), *\right)$ is isomorphic to an algebra of operators, or, as equation (3.78) suggests, to an algebra of infinite dimensional matrices.

3.2.2 A sequence of non abelian algebras

A fuzzy space has been presented as a sequence of finite rank matrix algebras converging, as compact quantum metric spaces, to an algebra of functions. In the case of the fuzzy sphere the rank of the matrices involved is the dimension of the Hilbert space on which UIRR's of the group $SU(2)$ are realised. In the approach sketched in the last section, based on a definition of a noncommutative plane, there is no natural, say intrinsic, definition of a set of finite dimensional matrix algebras. This, following the general comment already expressed, can be thought of as a consequence of the fact that also this noncommutative plane has been realised via a Berezin quantization based on coherent states originated by a group, the Heisenberg-Weyl, which is noncompact.

In this context the strategy to obtain finite dimensional matrix algebras is different. $\mathcal{A}_\theta$ can be considered, once a basis in the Hilbert space has been chosen, as a matrix algebra, formally made up of, infinite dimensional matrices. One can define a set of finite dimensional matrix algebras simply truncating $\mathcal{A}_\theta$. The notion of truncation is formalised via the introduction of a set of projectors in the space of operators. Their symbols are projectors in the algebra $\mathcal{A}_\theta$ of the noncommutative plane, in the sense that they are idempotent functions of order 2 with respect to the Voros product (here $z = re^{i\varphi}$):

$$\begin{aligned} P_\theta^{(N)}(r, \varphi) &= \sum_{n=0}^{N} \langle z \mid \psi_n \rangle \langle \psi_n \mid z \rangle = e^{-r^2/\theta} \sum_{n=0}^{N} \frac{r^{2n}}{n!\theta^n} \\ P_\theta^{(N)} * P_\theta^{(N)} &= P_\theta^{(N)} \end{aligned} \tag{3.79}$$

This finite sum can be performed yielding a rotationally symmetric function:

$$P_\theta^{(N)}(r, \varphi) = \frac{\Gamma\left(N+1, r^2/\theta\right)}{\Gamma(N+1)} \tag{3.80}$$

in terms of the ratio of an incomplete gamma function by a gamma function [33]. If θ is kept fixed, and nonzero, in the limit for $N \to \infty$ the symbol $P_\theta^{(N)}(r, \varphi)$

converges, pointwise, to the constant function $P_\theta^{(N)}(r,\varphi) = 1$, which can be formally considered as the symbol of the identity operator: in this limit one recovers the "whole" noncommutative plane.

This situation changes if the limit for $N \to \infty$ is performed keeping the product $N\theta$ equals to a constant, say R^2. In a pointwise convergence, chosen $R^2 = 1$:

$$P_\theta^{(N)} \;\to\; \left[\begin{array}{cc} 1 & r<1 \\ 1/2 & r=1 \\ 0 & r>1 \end{array}\right] = Id\,(r) \qquad (3.81)$$

This sequence of projectors converges to a sort of step function in the radial coordinate r, or a characteristic function of a disc on the plane. The shape of this function is plotted in figure 3.1. Already for $N = 10^3$ it is well approximated

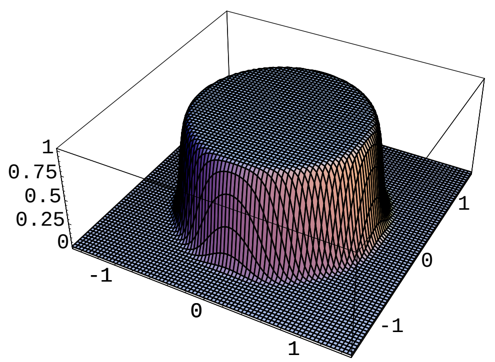

Figure 3.1: *The function $P_\theta^{(N)}$ for $N = 10^2$.*

(see figure 3.2) by a step function. Thus a sequence of subalgebras $\mathcal{A}_\theta^{(N)}$ can be

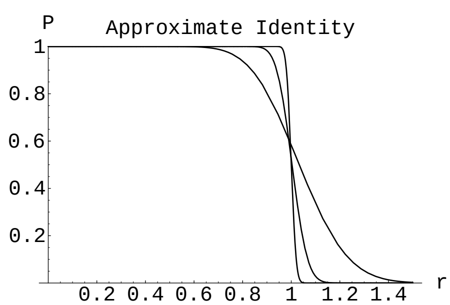

Figure 3.2: *Profile of the spherically symmetric function P_θ^N for the choice $R^2 = N\theta = 1$ and $N = 10, 10^2, 10^3$. As N increases the step becomes sharper.*

defined by:

$$\mathcal{A}_\theta^{(N)} = P_\theta^{(N)} * \mathcal{A}_\theta * P_\theta^{(N)} \qquad (3.82)$$

As it has been said, the full algebra $\mathcal{A}_\theta$ is isomorphic to an algebra of operators. What the previous relation says is that $\mathcal{A}_\theta^{(N)}$ is isomorphic to $\mathbb{M}_{N+1}$, the algebra of $(N+1)\times(N+1)$ rank matrices: the important thing is that this isomorphism (this truncation) is obtained via a specific choice of a basis in the Fock space $\mathcal{F}$ on which coherent states for the Heisenberg-Weyl group are realised. The explicit effect of this projection on a generic function is:

$$\Pi_\theta^{(N)}(f) \;=\; f_\theta^{(N)} \;=\; P_\theta^{(N)} * f * P_\theta^{(N)} \;=\; e^{-|z|^2/\theta} \sum_{m,n=0}^{N} f_{mn} \frac{\bar{z}^m z^n}{\sqrt{m!n!\theta^{m+n}}} \tag{3.83}$$

On every subalgebra $\mathcal{A}_\theta^{(N)}$, the symbol $P_\theta^{(N)}(r,\varphi)$ is then an identity, because it is the symbol of the projector $\hat{P}^{(N)} = \sum_{n=0}^{N} \mid \psi_n\rangle\langle\psi_n \mid$, which is the identity operator in $\mathcal{A}_\theta^{(N)}$, or, equivalently, the identity matrix in every $\mathbb{M}_{N+1}$.

It is important to note that the rotation group on the plane, $SO(2)$, acts in a natural way on these subalgebras. Its generator is the number operator $\hat{N} = \sum_{n=0}^{N} n\theta \mid \psi_n\rangle\langle\psi_n \mid$, that is diagonal in each $\mathcal{A}_\theta^{(N)}$.

Cutting at a finite N the expansion provides an infrared cutoff. This cutoff is "fuzzy" in the sense that functions in the subalgebra are still defined outside the disc of radius R, but are exponentially damped. To analyse the nature of this cutoff, it is interesting to study how this truncation works in a test case of a gaussian function, cylindrically symmetric and centred at the origin of the plane:

$$\Phi(r) \;=\; \frac{1}{\pi\alpha} e^{-r^2/\alpha} \tag{3.84}$$

The width of this gaussian is proportional to the parameter α. This function can be expanded in Taylor series:

$$\Phi(r) \;=\; \frac{1}{\pi\alpha} e^{-r^2/\alpha} \;=\; \frac{1}{\pi\alpha} \sum_{s=0}^{\infty} \left(-\frac{1}{\alpha}\right)^s \frac{1}{s!} \bar{z}^s z^s \tag{3.85}$$

Formulas (3.66) and (3.68) show that this function is mapped into the operator:

$$\hat{\Phi} \;=\; \frac{1}{\pi\alpha} \sum_{s=0}^{\infty} \left(-\frac{1}{\alpha}\right)^s \frac{1}{s!} \hat{a}^{\dagger\, s} \hat{a}^s \tag{3.86}$$

This operator is a diagonal operator, whose form is given by:

$$\hat{\Phi} \;=\; \frac{1}{\pi\alpha} \sum_{n=0}^{\infty} \left(1 - \frac{\theta}{\alpha}\right)^n \mid \psi_n\rangle\langle\psi_n \mid \tag{3.87}$$

The symbol of the truncated version is then:

$$\Pi_\theta^N(\Phi) \;=\; e^{-r^2/\theta} \sum_{n=0}^{N} \frac{1}{\pi\alpha} \left(1 - \frac{\theta}{\alpha}\right)^n \frac{r^{2n}}{\theta^n n!} \;=\; e^{-r^2/\alpha} \frac{\Gamma\left(N+1, r^2\left(1/\theta - 1/\alpha\right)\right)}{\pi\alpha\Gamma(N+1)} \tag{3.88}$$

This formula clearly shows that the behaviour of the projected function, for increasing N, is related to a comparison between the values of $\theta = 1/N$ and α.

A first interesting analysis on the nature of the projection is given by a direct computation:

$$
\begin{aligned}
\hat{\Phi}^{\dagger}\hat{\Phi} &= \left(\frac{1}{\alpha\pi}\right)^2 \sum_{n=0}^{\infty} \left(1-\frac{\theta}{\alpha}\right)^{2n} \mid \psi_n\rangle\langle\psi_n \mid \\
Tr\left[\hat{\Phi}^{\dagger}\hat{\Phi}\right] &= \left(\frac{1}{\alpha\pi}\right)^2 \sum_{n=0}^{\infty} \left(1-\frac{\theta}{\alpha}\right)^{2n} = \frac{1}{(\pi\alpha)^2} \lim_{n\to\infty} \frac{1-(1-\theta/\alpha)^{2(n+1)}}{1-(1-\theta/\alpha)^2}
\end{aligned}
\tag{3.89}
$$

The limit in the RHS of this relation is finite if $|1-\theta/\alpha|^2 < 1$:

- for $\alpha > \theta$, $\hat{\Phi}$ is an Hilbert-Schmidt operator, so the projection gives a sequence, for $N \to \infty$, of finite rank matrices $\hat{\Phi}_\theta^{(N)}$ converging in the strong operator topology to $\hat{\Phi}$. This means that the sequence of symbols $\Phi_\theta^{(N)}(r) = \Pi_\theta^{(N)}(\Phi)$ converges (pointwise) to a function which is equal to $\Phi(r)$ inside the disc of radius 1, and equal to zero outside the disc. The plot is in figure 3.3.

- for $\alpha = \theta$ the projection is trivial, as in this case the operator $\hat{\Phi}$ is just a multiple of the projector:

$$
\hat{\Phi}_\theta^{(N)} = \frac{1}{\pi\theta} \mid \psi_0\rangle\langle\psi_0 \mid
$$

- for $\alpha < \theta < 2\alpha$ the operator $\hat{\Phi}$ is still Hilbert-Schmidt, so the sequence of truncated operators converges to the operator $\hat{\Phi}$ in the strong operator topology, and the sequence of projected symbols converges again to the symbol $\Phi(r)$ inside the disc, and to zero outside.

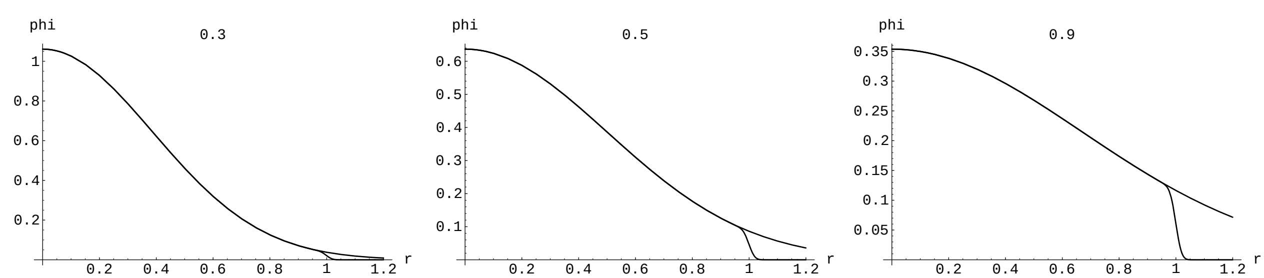

Figure 3.3: *Profile of the spherically symmetric function* $\Pi_\theta^N\left(\frac{1}{\pi\alpha}e^{-\frac{r^2}{\alpha}}\right)$ *for the choice* $R^2 = N\theta = 1$. *Here* $N = 10^3$, *so* $\theta = 0.001$, *and* α *has been chosen to be the value on the top of each plot. For these three choices* α *is much larger than* θ. *Both the projected and the unprojected functions are plotted, although inside the disc they are practically indistinguishable. The unprojected function is always the larger one.*

The operator $\hat{\Phi}$ is no more of the Hilbert-Schmidt class starting from $\theta = 2\alpha$, and for $\theta > 2\alpha$ it is no more compact. That something is happening to the sequence of projected symbols is evident by figure 3.4

With $\alpha = .5\theta$ a small "bump" at $r = 1$ appears. Already for $\alpha = .49\theta$ the "bump" has become a large gaussian sitting at the infrared cutoff; the part close to the origin is still present, but it is quickly dwarfed by this bump, growing very fast: in a numerical approximation, for $\alpha \sim .4\theta$ it is already of the order of 10^{17}. In this case the series expansion (3.89) is alternating, and individual terms are divergent. A very interesting point is to stress: keeping α fixed, and increasing N with the introduced constraint $N\theta = 1$, the bump disappears. This limit forces in fact θ to go to 0, thus recovering the "nice" behaviour.

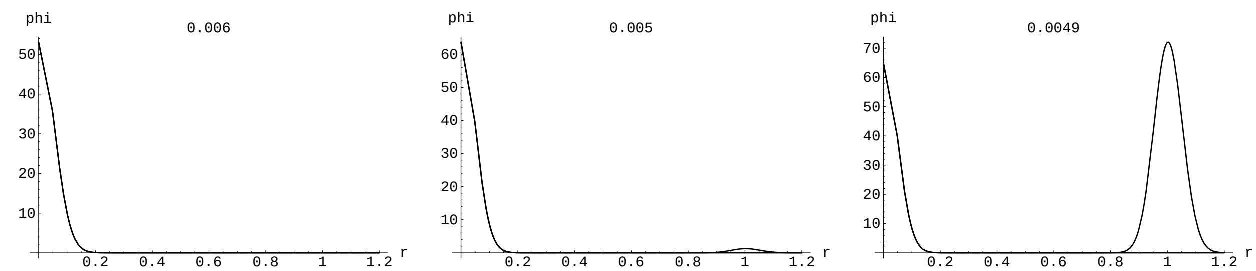

Figure 3.4: *Profile of the spherically symmetric function* $\Pi_\theta^N\left(\frac{1}{\pi\alpha}e^{-\frac{r^2}{\alpha}}\right)$ *for the choice* $R^2 = N\theta = 1$, $N = 10^2$. *Here* $\theta = 0.01$ *and the value of* α *is chosen to be around the "turning point"* $\alpha = \theta/2$.

This detailed analysis shows that the Weyl map used to define this quantization (3.66) is very different from the standard Weyl map described in the first chapter (1.47). In the standard case it has been proved that it is an isomorphism between square integrable functions on the plane, and Hilbert-Schmidt class operators. In this case, it is evident that the role of the weight factor in (3.74) is to change the set of applicability of this correspondence.

In general the function Φ is close to its projected version $\Phi_\theta^{(N)}$ if it is mostly supported on a disc of radius 1 (of radius $R = \sqrt{N\theta}$, in general), otherwise it is simply exponentially cut, and if it has not oscillations of too small wavelenght (compared to θ). In this case the projected function becomes very large on the boudary of the disc. This can be seen as a compact example of the *ultraviolet-infrared* mixing, which is one of the most interesting characteristic of noncommutative geometry [26]. If one tries to localise the function too much, unavoidably an infrared divergence on the boundary of the disc appears.

There are however functions which are localised sharply near the edge of the disc. These can be seen as edge states [32], which play an important role in Chern-Simons theory. These edge states are given, in each finite rank approximation, by the symbols of the highest one dimensional projectors:

$$\phi^{edge} \equiv \frac{1}{\theta}\langle z \mid \psi_N\rangle\langle\psi_N \mid z\rangle = e^{-r^2/\theta}\,\frac{r^{2N}}{N!\theta^{N+1}} \tag{3.90}$$

They are plotted in figure 3.5.

3.2.3 Fuzzy derivatives

So far the Weyl-Wigner formalism, and the projection procedure, have provided a way to associate to functions on the plane a sequence of finite dimension

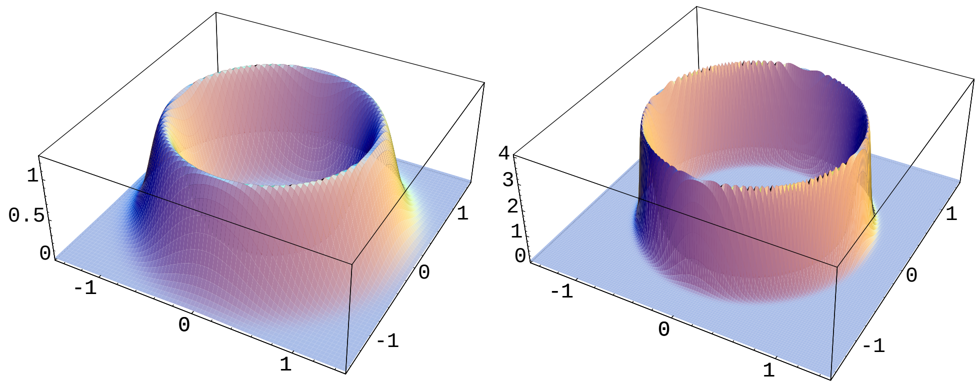

Figure 3.5: *The edge states* φ^{edge} *for* $N = 10$ *and* $N = 100$.

$(N+1) \times (N+1)$ matrices. An appropriate choice of θ, the noncommutativity parameter introduced by the quantization map, and N, showed that it is possible to obtain a good approximation of a certain class of functions supported on a disc.

The next step is the analysis of the geometry this algebras can formalize. To pursue this task, it is important to define, in this sequence of algebras, both derivations and a Laplacian. The starting point to define the matrix equivalent of the derivations is:

$$\begin{aligned} \partial_z f &= \frac{1}{\theta} \langle z \mid \left[\hat{f}, \hat{a}^\dagger \right] \mid z \rangle \\ \partial_{\bar{z}} f &= \frac{1}{\theta} \langle z \mid \left[\hat{a}, \hat{f} \right] \mid z \rangle \end{aligned} \tag{3.91}$$

This relation is exact in the full algebra $\mathcal{A}_\theta$. Given an operator $\hat{f}$, the derivatives of the symbol $f(\bar{z}, z)$ are related to the symbol of the commutator of $\hat{f}$ with the creation and annihilation operators. A fuzzyfied version, namely a truncated version, of these operations, can be introduced defining:

$$\begin{aligned} \partial_z f_\theta^{(N)} &\equiv \frac{1}{\theta} \langle z \mid \hat{P}_\theta^{(N)} \left[\hat{P}_\theta^{(N)} \hat{f} \hat{P}_\theta^{(N)}, \hat{a}^\dagger \right] \hat{P}_\theta^{(N)} \mid z \rangle \\ \partial_{\bar{z}} f_\theta^{(N)} &\equiv -\frac{1}{\theta} \langle z \mid \hat{P}_\theta^{(N)} \left[\hat{P}_\theta^{(N)} \hat{f} \hat{P}_\theta^{(N)}, \hat{a} \right] \hat{P}_\theta^{(N)} \mid z \rangle \end{aligned} \tag{3.92}$$

It is important to note that this is actually a derivation (it is a linear operation, satisfying the Leibniz rule) on each $\mathcal{A}_\theta^{(N)}$. The idea behind this definition is to consider an element $\hat{f}_\theta^{(N)}$ of a finite step of the sequence as a finite rank matrix in a space of infinite dimensional matrices. The projection is then seen as an embedding of the "truncated" matrix $\hat{f}_\theta^{(N)}$ as an upper left block diagonal matrix into an infinite dimensional matrix where the remaining infinite number of rows and columns have elements equal to zero. The way this embedding acts is in the left hand term of the inner commutator: $\hat{P}_\theta^{(N)} \hat{f} \hat{P}_\theta^{(N)}$. Next the "exact" relation (3.91) for derivatives of operators in $\mathcal{F}$ - so to say infinite matrices, in this context - is used. This is the meaning of the commutator with operators $\hat{a}$

and $\hat{a}^\dagger$, which are considered in their complete matrix representation, without any "truncation". Last, since these derivations should map dimensional matrix of $\mathcal{A}_\theta^{(N)}$ into a finite dimensional matrix in the same $\mathcal{A}_\theta^{(N)}$, the image of the commutator should be projected back to the finite dimensional space generated by the first $N+1$ ket state $\mid\psi_n\rangle$. Moreover, this projection is important, because creation and annihilation operators are ladder operators, and then they tend to shift N^{th} column's and row's element into the $(N+1)^{th}$ column and row of the image matrix.

The first example is the calculation of the derivatives of the fuzzyfied coordinate functions z and $\bar{z}$. From the definition, z is the symbol of the annihilation operator, while $\bar{z}$ is the symbol of the creation operator, that can be written as:

$$\begin{aligned} \hat{a} &= \sum_{s=0}^{\infty} \sqrt{(s+1)\,\theta} \mid \psi_s\rangle\langle\psi_{s+1} \mid \\ \hat{a}^\dagger &= \sum_{k=0}^{\infty} \sqrt{(k+1)\,\theta} \mid \psi_{k+1}\rangle\langle\psi_k \mid \end{aligned} \tag{3.93}$$

Projection into $\mathcal{A}_\theta^{(N)}$ gives:

$$\hat{a}_\theta^{(N)} = \sum_{s=0}^{N-1} \sqrt{(s+1)\,\theta} \mid \psi_s\rangle\langle\psi_{s+1} \mid \tag{3.94}$$

To perform the derivative with respect to z variable, one considers:

$$\begin{aligned} \left[\hat{a}_\theta^{(N)}, \hat{a}^\dagger\right] &= \theta\left[\mid\psi_0\rangle\langle\psi_0\mid + \sum_{s=1}^{N-1} \mid\psi_s\rangle\langle\psi_s\mid - N\mid\psi_N\rangle\langle\psi_N\mid\right] \\ &= \theta\left[\sum_{s=0}^{N-1} \mid\psi_s\rangle\langle\psi_s\mid - N\mid\psi_N\rangle\langle\psi_N\mid\right] \\ &= \theta\left[\sum_{s=0}^{N} \mid\psi_s\rangle\langle\psi_s\mid - (1+N)\mid\psi_N\rangle\langle\psi_N\mid\right] \end{aligned} \tag{3.95}$$

The first term of the sum is the projector onto the first $N+1$ basis elements, the identity on $\mathcal{A}_\theta^{(N)}$, a fuzzy identity. What is interesting is that this commutator has no terms 'outside' the space we are considering, namely there are no components on density matrices of order greater than N: this means that in this case there is no need to project it on $\mathcal{A}_\theta^{(N)}$. The symbol of this commutator is:

$$\partial_z\left(z_\theta^{(N)}\right) = e^{-Nr^2}\left[\sum_{s=0}^{N} \frac{r^{2s}N^s}{s!} - \frac{N+1}{N!} r^{2N} N^N\right] \tag{3.96}$$

In the limit of $N \to \infty$ the first term is what has been called 'characteristic function for the disc', while the second converges to a factor $\pi\delta\,(r-1)$. This factor is a radial δ selecting the value for $r = 1$ with respect to the Lebesgue measure on the plane:

$$\lim_{N\to\infty} \partial_z\left(z_\theta^{(N)}\right) = Id\,(r) - \pi\delta\,(r-1) \tag{3.97}$$

To calculate the derivative of $\bar{z}$ with respect to z one needs to consider:

$$\hat{a}_\theta^{\dagger(N)} = \sum_{k=0}^{N-1} \sqrt{(k+1)\,\theta} \mid \psi_{k+1}\rangle\langle\psi_k \mid$$

obtaining:

$$\left[\hat{a}_\theta^{\dagger(N)}, \hat{a}^\dagger\right] = -\theta\sqrt{N\,(N+1)} \mid \psi_{N+1}\rangle\langle\psi_N \mid \tag{3.98}$$

Since this operator must be projected back to the algebra $\mathcal{A}_\theta^{(N)}$, one finally has:

$$\partial_{\bar{z}}\left(z_\theta^{(N)}\right) = 0 \tag{3.99}$$

3.2.4 Fuzzy Laplacian and fuzzy Bessels

One can consider the problem for the Laplacian in a similar context. From the exact expressions:

$$\begin{aligned}\nabla^2 f &= 4\partial_{\bar{z}}\partial_z f \\ \left(\nabla^2 f\right)(\bar{z}, z) &= \frac{4}{\theta^2}\langle z \mid \left[\hat{a}, \left[\hat{f}, \hat{a}^\dagger\right]\right] \mid z\rangle\end{aligned} \tag{3.100}$$

it is possible to define, in each $\mathcal{A}_\theta^{(N)}$:

$$\nabla^2 \hat{f}_\theta^{(N)} \equiv \frac{4}{\theta^2}\hat{P}_\theta^{(N)}\left[\hat{a}, \left[\hat{P}_\theta^{(N)}\hat{f}\hat{P}_\theta^{(N)}, \hat{a}^\dagger\right]\right]\hat{P}_\theta^{(N)} \tag{3.101}$$

The image of the element of the truncated algebra:

$$\hat{f}_\theta^{(N)} = \sum_{a,b=0}^{N} f_{ab} \mid \psi_a\rangle\langle\psi_b \mid$$

is:

$$\begin{aligned}\nabla^2 f_\theta^{(N)} &= 4N\left[\sum_{s=0}^{N-1}\sum_{b=0}^{N-1} f_{s+1,b+1}\sqrt{(s+1)\,(b+1)} \mid \psi_s\rangle\langle\psi_b \mid + \right.\\ &- \sum_{s=0}^{N}\sum_{b=0}^{N} f_{sb}\,(s+1) \mid \psi_s\rangle\langle\psi_b \mid - \sum_{s=0}^{N-1} f_{0,s+1}\,(s+1) \mid \psi_0\rangle\langle\psi_{s+1} \mid + \\ &+ \sum_{s=0}^{N-1}\sum_{b=0}^{N-1} f_{sb}\sqrt{(s+1)\,(b+1)} \mid \psi_{s+1}\rangle\langle\psi_{b+1} \mid + \\ &\left. - \sum_{s=0}^{N-1}\sum_{b=0}^{N-1} f_{s+1,b+1}\,(b+1) \mid \psi_{s+1}\rangle\langle\psi_{b+1} \mid \right]\end{aligned} \tag{3.102}$$

The spectrum of this *fuzzy Laplacian* can be numerically calculated. Its eigenvalues seem to converge to the spectrum of the continuum Laplacian for functions on a disc, with boundary conditions on the edge of the disc of Dirichlet homogeneous kind [40]. In the continuum case, the eigenvalue problem for the Laplacian with Dirichlet homogeneous boundary conditions is solved by the zeroes of the Bessel functions of first kind, namely λ is an eigenvalue if it solves the implicit equation

$$J_n\left(\sqrt{\lambda}\right)=0 \tag{3.103}$$

where n is the order of the Bessel. In particular, those related to J_0 are simply degenerate, the other are doubly degenerate: so there is a sequence of eigenvalues labelled by $\lambda_{n,k}$ where n is the order of the Bessel function and k indicates that it is the k^{th} zero of the function. The spectrum of the fuzzy Laplacian is in good agreement with the spectrum of the continuum explained case, even for low values N of the dimension of truncation, as can be seen in figure 3.6.

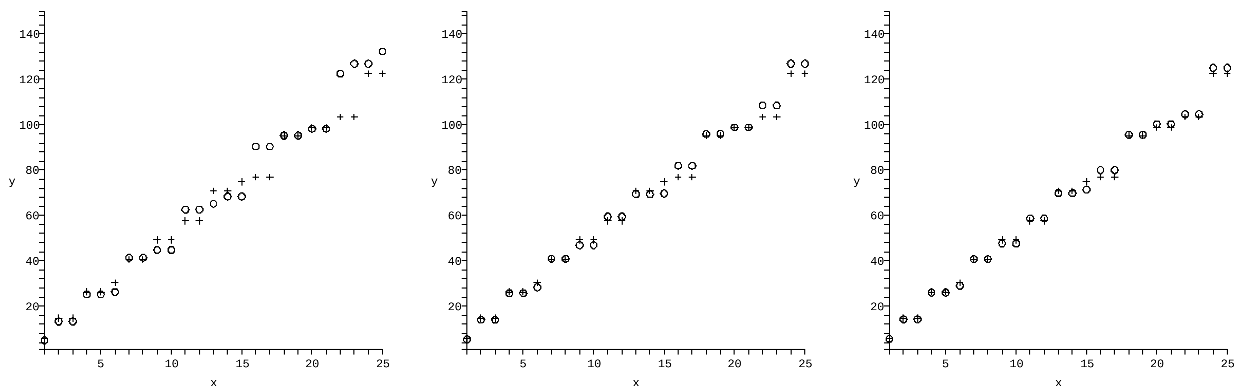

Figure 3.6: *Comparison of the first eigenvalues of the fuzzy Laplacian (circles) with those of the continuum Laplacian (crosses) on the domain of functions with Dirichlet homogeneous boundary conditions. The orders of truncation are* $N=9,19,29$.

What is interesting is that this definition of Laplacian (3.101) gives the expected pattern of non degenerate and double degenerate eigenvalues. The difference with the case of the spectrum of the fuzzy laplacian for the fuzzy sphere is that now the "fuzzy spectrum" is both a cut-off and an approximation of the continuum spectrum. It is a cut-off because, of course, it is a finite rank operator. The fact that it is an approximation is related to the fact that it has been defined using a formalism whose building blocks are related to a noncompact group, namely the Heisenberg-Weyl, so that there is no finite dimensional realization of its generators.

The eigenfunctions for the continuum problem are:

$$\psi_{n,k}=e^{in\varphi}\left(\frac{\sqrt{\lambda_{|n|,k}}r}{2}\right)^{|n|}\sum_{k=0}^{\infty}\frac{\left(-\lambda_{|n|,k}\right)^k}{k!\left(|n|+k\right)!}\left(\frac{r}{2}\right)^{2k}=e^{in\varphi}J_{|n|}\left(\sqrt{\lambda_{|n|,k}}r\right) \tag{3.104}$$

In this expression n is an integer number, $|n|$ is its absolute value: this is a way to write eigenfunctions in a compact form, taking into account the degeneracy of eigenvalues for $|n|\geq 1$. Now, in the 'fuzzy' approximation, one has a sequence of eigenvalues $\lambda^{(N)}_{n,k}$, with N indicating the dimension of the fuzzyfication. The label

n runs from $-N$ to $+N$, while k runs from 1 to $N-|n|+1$. To each eigenvalue one can associate an eigenmatrix, indicated with $\hat{\Phi}\left(\lambda_{n,k}^{(N)}\right)$. Its symbol is a function of $z, \bar{z}$:

$$\Phi\left(\lambda_{n,k}^{(N)}\right) = e^{-N|z|^2} \sum_{a,b=0}^{N} \Phi_{ab}^{(N)} \left(\lambda_{n,k}^{(N)}\right) \bar{z}^a z^b \left(\frac{N^{a+b}}{a!b!}\right)^{1/2} \tag{3.105}$$

The expansion of these symbols, those of the eigenmatrices, in Taylor series, are seen to coincide, up to order N, with the Taylor expansion of the eigenfunctions of the continuum problem. This is the reason why the eigenmatrices can be defined as *fuzzy Bessels.*

Some of them are plotted. In figure 3.7 it is plotted the radial shape of the fuzzy Bessel relative to $n=0$ and $k=1$, so to say the fuzzy ground state of this matrix model. The comparison shows that this fuzzy ground state converges to the continuum eigenfunctions $\psi_{0,1}(r,\varphi)$ for values of r inside the disc of radius 1, while it converges to zero outside the disc.

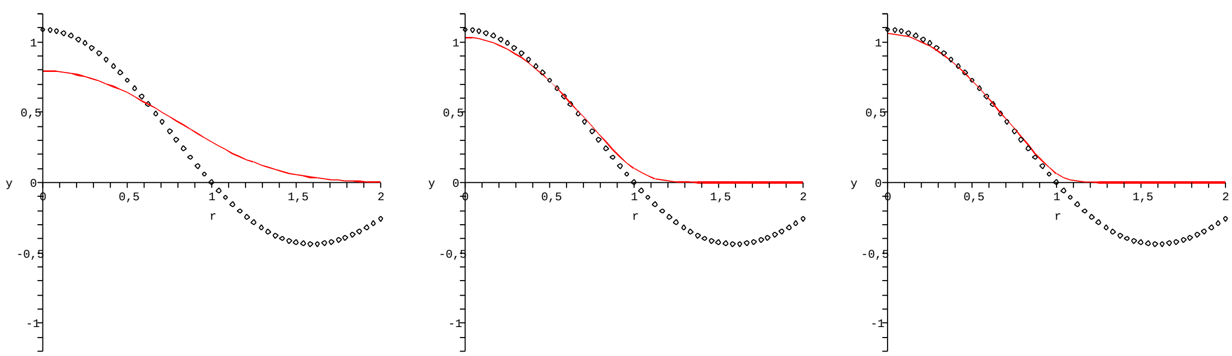

Figure 3.7: *Comparison of the radial shape for the ground state fuzzy Bessel* $\Phi\left(\lambda_{0,1}^{(N)}\right)$ *(continuum line), the symbol of the eigenmatrix of the fuzzy Laplacian with respect to the lowest eigenvalue, with* $\psi_{0,1}(r,\varphi)$*, the ground state eigenfunction of the continuum problem (diamond line), for* $N=3,15,30$*. The fuzzy Bessel converges to zero outside the disc of radius* 1.

This behaviour is valid also for eigenstates of different eigenvalues. The plots are for the symbols $\Phi\left(\lambda_{0,2}^{(N)}\right)$ in figure 3.8 and $\Phi\left(\lambda_{0,3}^{(N)}\right)$ in figure 3.9. Symbols $\Phi\left(\lambda_{0,k}^{(N)}\right)$ and functions $\psi_{0,k}(r,\varphi)$ are seen to be radial functions.

Since fuzzy Bessels are obtained as eigenstates of an Hermitian operator, they are defined up to a normalization factor. In view of their use, the continuum Bessel functions are normalised as:

$$\int_0^{2\pi} d\varphi \int_0^1 r\, dr \left| J_{|n|}\left(\sqrt{\lambda_{|n|,k}}\, r\right)\right|^2 = 1 \tag{3.106}$$

while fuzzy Bessels are normalised by:

$$\int_0^{2\pi} d\varphi \int_0^{\infty} r\,dr \left|\Phi\left(\lambda_{|n|,k}^{(N)}\right)\right|^2 = 1 \tag{3.107}$$

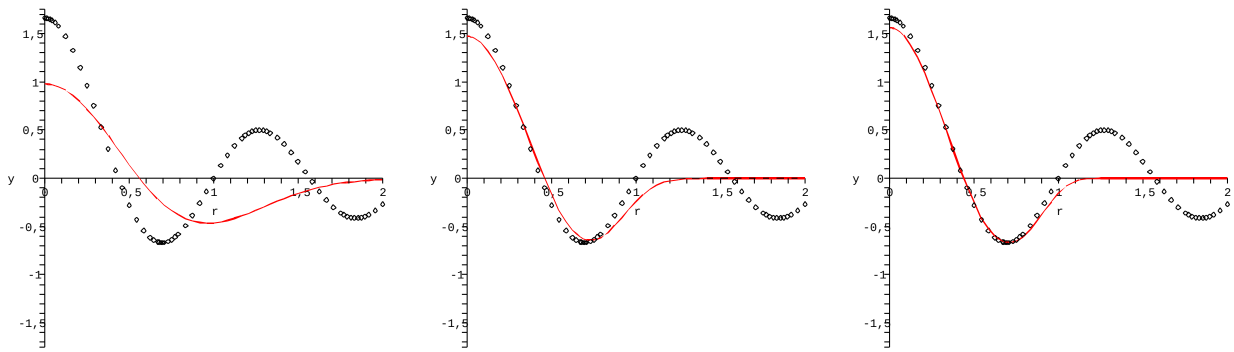

Figure 3.8: *Comparison of the radial shape for the* $\Phi\left(\lambda_{0,2}^{(N)}\right)$ *fuzzy Bessel (continuum line), the symbol of the eigenmatrix of the fuzzy Laplacian with respect to the eigenvalue* $\lambda_{0,2}^{(N)}$*, with the function* $\psi_{0,2}(r)(r,\varphi)$ *that is the eigenfunction of the continuum problem with eigenvalue* $\lambda_{0,2}$*. Here the orders of truncation are* $N = 2, 15, 30$*. The fuzzy Bessel converges to zero outside the disc of radius* 1.

These functions will play a role similar to the role 'fuzzy harmonics' play in the fuzzy sphere algebra: they will be seen as a fuzzy version of the basis of the space of functions on the disc.

Given a function on the disc, if it is square integrable, $f \in \mathcal{L}^2(D, r dr d\varphi)$:

$$f(r,\varphi) = \sum_{n=-\infty}^{+\infty} \sum_{k=1}^{\infty} f_{nk} e^{in\varphi} J_{|n|}\left(\sqrt{\lambda_{|n|,k}}\,r\right) \tag{3.108}$$

it is possible to truncate:

$$f^{(N)}(r,\varphi) = \sum_{n=-N}^{+N} \sum_{k=1}^{N+1-|n|} f_{nk} e^{in\varphi} J_{|n|}\left(\sqrt{\lambda_{|n|,k}}\,r\right) \tag{3.109}$$

This set of functions is a vector space, but it is no more an algebra with respect to the pointwise product. The mapping

$$\hat{f}_\theta^{(N)} = \sum_{n=-N}^{+N} \sum_{k=1}^{N+1-|n|} f_{nk} \hat{\Phi}\left(\lambda_{n,k}^{(N)}\right) \tag{3.110}$$

define a sequence (indexed by N) of finite rank matrix algebras, whose formal limit is an abelian algebra because of the constraint $N\theta = 1$. This is defined a *fuzzy disc.*

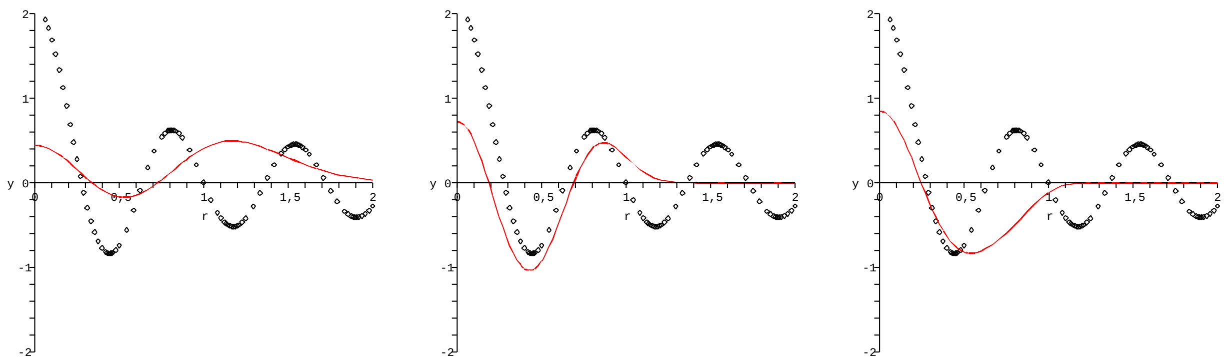

Figure 3.9: *Comparison of the radial shape for the* $\Phi\left(\lambda_{0,3}^{(N)}\right)$ *fuzzy Bessel (continuum line), the symbol of the eigenmatrix of the fuzzy laplacian with respect to the eigenvalue* $\lambda_{0,3}^{(N)}$*, with the function* $\psi_{0,3}\left(r,\varphi\right)$ *that is the eigenfunction of the continuum problem with eigenvalue* $\lambda_{0,3}$*. Here the orders of truncation are* $N=2,15,30$*. The fuzzy Bessel converges to zero outside the disc of radius* 1*.*

3.2.5 Free Field Theory on the Fuzzy Disc: Green's functions

The formalism developed lends itself readily for matrix approximations to field theories on a disc [5]. For the real scalar case described by the action:

$$S=\int d^2z\ \phi\nabla^2\phi+\frac{m^2}{2}\phi^2+V(\phi) \tag{3.111}$$

a quantum version can be studied using the path integral formalism. This path integral is ill defined, relying on the concept of an integral with an infinite dimensional functional measure. But the path integral can be made rigorous if the space of admissible configurations for field variables is made finite. Such is the case with this approximation. The fuzzy version of the action is:

$$S_\theta^{(N)}=\frac{1}{\pi}Tr\left[\hat\Phi_\theta^{(N)}\nabla^2\hat\Phi_\theta^{(N)}+\frac{m^2}{2}\hat\Phi_\theta^{(N)}\cdot\hat\Phi_\theta^{(N)}+V(\hat\Phi_\theta^{(N)})\right] \tag{3.112}$$

A first analysis is the calculation of the two points Green function for the free massless scalar theory. In this case the path integral may be explicitly performed, yielding just the inverse of the Laplacian with Dirichlet boundary conditions which has been already computed. The fuzzy Laplacian is a map from $\mathcal{A}_\theta^{(N)}$ to itself, so it can be seen as a linear operator acting on the space $\mathbb{C}^{(N+1)^2}$, so to say a matrix belonging to $\mathbb{M}_{(N+1)^2}$. Its inverse will be a matrix belonging to the same space, that can be mapped into a two points symbol via:

$$G_\theta^{(N)}(z,z')=4\sum_{m,n,p,q=0}^{N}\frac{e^{-\frac{|z|^2+|z'|^2}{\theta}}(\nabla^2)^{-1}_{mnpq}\bar z^p z^q z'^m\bar z'^n}{\sqrt{p!q!m!n!\theta^{m+n+p+q}}} \tag{3.113}$$

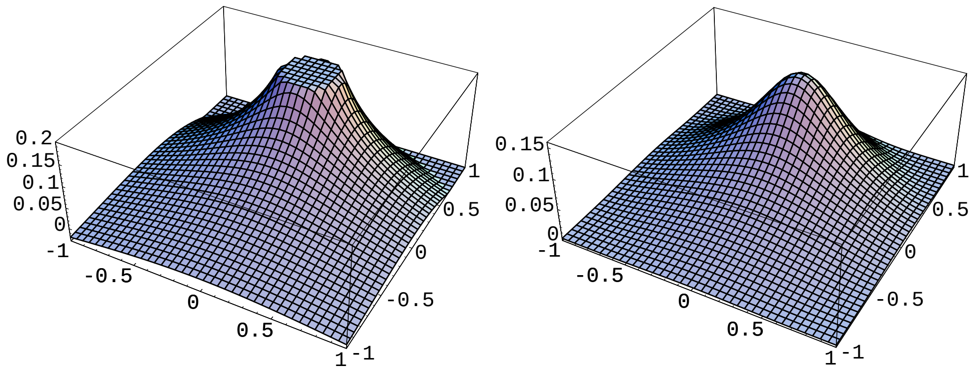

Figure 3.10: *Comparison between the Green's function $G(z, z')$ as function of z for $z' = i/2$ On the left the exact function (with the singularity at $z = z'$ truncated), on the right the approximated function for $N = 20$.*

This expression can be evaluated numerically (figure 3.10) and compared with the exact case, known from the classic theory of electrodynamics:

$$G(z, z') = \frac{1}{2\pi} \ln \frac{|z - z'|}{|z'|z| - z|z|^{-1}|} \tag{3.114}$$

The agreement is quite remarkable already for a small value of N. The logarithmic divergence has been smoothened by the ultraviolet cutoff, but apart from that the two functions are quite similar. The choice of different values of z' gives similar pictures.

It is very interesting to note that, since fuzzy Laplacian is represented by an hermitian finite dimensional matrix, its inverse can be written in terms of a spectral decomposition. If a matrix, say M, is hermitian, then its components satisfy the condition $M_{ab} = M^*_{ba}$ in terms of transposition and complex conjugation. Eigenvalue problem gives a number of real eigenvalues equal to the dimension of the space on which the matrix acts:

$$\sum_b M_{ab} v_b^{(k)} = \lambda^{(k)} v_a^{(k)} \tag{3.115}$$

here $v_a^{(k)}$ indicates the a^{th} components of the eigenvector relative to the eigenvalue $\lambda^{(k)}$. The inverse, if it exists, of the matrix M is a matrix G whose components can be written as:

$$G_{sq} = \sum_k v_s^{(k)} v_q^{(k)} / \lambda^{(k)} \tag{3.116}$$

This line can be translated in the specific problem under analysis. The notion of eigenvector of components $v_a^{(N)}$ with eigenvalue $\lambda^{(k)}$ goes into that of fuzzy Bessel $\widehat{\Phi}\left(\lambda_{n,k}^{(N)}\right)$ matrix, from which it is immediate to obtain the symbols. The fuzzy Green function becomes:

$$G_\theta^{(N)}(z, z') = \sum_{n=-N}^{+N} \sum_{k=1}^{N+1-|n|} \frac{\psi_{n,k}^{(N)}(z')^* \psi_{n,k}^{(N)}(z)}{\lambda_{|n|,k}^{(N)}} \tag{3.117}$$

This expression is exactly the same obtained by Madore in the case of a field theory on a fuzzy sphere: the role of fuzzy harmonics is now played by fuzzy Bessels.

Appendix A

A.1 An elementary introduction to the theory of C^*-algebras

This appendix is meant to be an introduction to some algebraic notions mentioned in the text. All the topics here recollected are covered, in more complete form, in [20] and [9].

Let $\mathcal{V}$ a vector space defined over $\mathbb{C}$, the field of complex numbers. A norm on the vector space $\mathcal{V}$ is a map:

$$\| \cdot \| : \mathcal{V} \mapsto \mathbb{R}$$

such that:

- $\| v \| \geq 0 \ \ \forall v \in \mathcal{V}; \| v \| = 0 \Leftrightarrow v = 0$
- $\| \lambda v \| = |\lambda| \ \| v \|; \forall \lambda \in \mathbb{C}$ and $v \in \mathbb{C}$
- $\| v + u \| \leq \| v \| + \| u \|$ (triangle inequality)

A norm on $\mathcal{V}$ defines a metric d on $\mathcal{V}$ by $d(v,u) \equiv \| v - u \|$. A vector space with a norm which is complete in the associated metric (in the sense that every Cauchy sequence converges) is called a Banach space. On a Banach space B a functional is a linear map:

$$\rho : \mathrm{B} \mapsto \mathbb{C}$$

The norm of such a functional is defined by:

$$\| \rho \| \equiv \sup \{ |\rho(v)| / \| v \| ; v \in \mathrm{B} \} \tag{A.1}$$

A linear functional is continuous if and only if it is bounded. The dual B^* space of a Banach space is the space of all functionals on B: it is an example of a Banach space.

On a vector space it is possible to define an inner product, a map:

$$\langle \cdot \mid \cdot \rangle : \mathcal{V} \times \mathcal{V} \mapsto \mathbb{C}$$

such that:

- it is bilinear in both entries
- $\langle v \mid v \rangle \geq 0 \ \forall v \in \mathcal{V}$. Moreover $\langle v \mid v \rangle = 0 \Leftrightarrow v = 0$

- it is skew-hermitian[1] : $\overline{\langle v \mid u \rangle} = \langle u \mid v \rangle \ \forall u \ v \in \mathcal{V}$

From these axioms, it is possible to derive the Cauchy-Schwarz inequality:

$$|\langle v \mid u \rangle|^2 \leq \langle v \mid v \rangle \langle u \mid u \rangle$$

An Hilbert space $\mathcal{H}$ is a vector space with an inner product which is complete in the associated norm: it is an example of a Banach space. It can be proved to be completely characterised by its dimension, i.e. the cardinality of an arbitrary orthogonal basis.

On a Banach space B a bounded operator is a linear map:

$$\hat{A} : \mathrm{B} \mapsto \mathrm{B}$$

whose norm is defined by:

$$\| \hat{A} \| \equiv \sup \{ \| \hat{A} v \| \, / \, \| v \| \, ; \, v \in \mathrm{B} \} \tag{A.2}$$

The space $\mathcal{B}(\mathrm{B})$ of bounded operators on a Banach space, with this definition of norm, is itself a Banach space.

An algebra is a vector space with an associative bilinear operation, the multiplication, which is compatible with the linear structure. If AB denotes such a multiplication:

$$(A + B) C = AC + BC \quad A (B + C) = AB + AC \tag{A.3}$$

A Banach space is defined to be a Banach algebra if it is an algebra such that the multiplication is separately continuous in each variable:

$$\| AB \| \leq \| A \| \| B \| \tag{A.4}$$

The Banach space of bounded operators on a Banach space $\mathcal{B}(\mathrm{B})$ is equipped with such a multiplication considering the product as the standard composition of operators: so it becomes a Banach algebra.

An involution on an algebra is a real-linear map $A \mapsto A^*$, satisfying, for every pair of algebra elements and every complex scalar:

- $A^{**} = A$
- $(AB)^* = B^* A^*$
- $(\lambda A)^* = \overline{\lambda} A^*$

A $*$-algebra is an algebra with an involution. The Banach algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space is a $*$-algebra if the involution is the operation $\hat{A} \mapsto \hat{A}^\dagger$ that maps the operator $\hat{A}$ into its adjoint $\hat{A}^\dagger$. Moreover, for this $*$-algebra there is a relation between the involution and the notion of norm, that is the operator norm. It can be proved that:

$$\| \hat{A} \| = \| \hat{A}^\dagger \| \tag{A.5}$$

The importance of a relation between the notion of involution and that of norm in a Banach algebra is such to motivate the introduction of the notion of C^*-algebra.

[1] The overline on the complex numbers indicates the complex conjugation.

A C^*-algebra is a Banach, complex, $*$-algebra, whose norm satisfies, for every pair of elements:

$$\begin{aligned} \| AB \| &\leq \| A \| \| B \| \\ \| AA^* \| &= \| A \|^2 \end{aligned} \tag{A.6}$$

The algebra of bounded operators on a Hilbert space is a C^*-algebra.

A morphism between two C^*-algebras $\mathcal{A}$ and $\mathcal{A}'$ is a complex-linear map $\phi : \mathcal{A} \mapsto \mathcal{A}'$ such that $\forall\, A\,,\, B \in \mathcal{A}$:

$$\begin{aligned} \phi\,(AB) &= \phi\,(A)\,\phi\,(B) \\ \phi\,(A^*) &= (\phi\,(A))^* \end{aligned} \tag{A.7}$$

An isomorphism is a bijective morphism. Two C^*-algebras are isomorphic if there is an isomorphism between them.

A unit in a C^*-algebra (or in a Banach algebra) is an element $\mathbf{1}$ such that, for every element A of this algebra one has:

$$A = A\mathbf{1} = \mathbf{1}A$$

For a C^*-algebra[2] this also implies that the norm of the identity is 1. With a unit, an algebra is called unital: an important result in this context is that a nonunital Banach algebra can always be extended to a unital Banach algebra.

A state on a C^*-algebra $\mathcal{A}$ is a linear map $\sigma : \mathcal{A} \mapsto \mathbb{C}$ which is positive (it means that $\forall\, A \in \mathcal{A}$, one has $\sigma\,(A^*A) \geq 0$) and of norm 1. The set of states $\mathcal{S}\,(\mathcal{A})$ is given the weak $*$-topology if the convergence of a sequence $\{\sigma_n\} \in \mathcal{S}\,(\mathcal{A})$ is defined by $\lim_n\, \sigma_n \to \sigma$ if and only if $\lim_n\, \sigma_n\,(A) = \sigma\,(A)$ for every $A \in \mathcal{A}$. Equipped with this topology, the set of states $\mathcal{S}\,(\mathcal{A})$ is a convex set. Pure states are those which cannot be expressed as a convex combination of states. If the C^*-algebra is unital, the set of states can be proved to be compact.

If X is a compact Hausdorff space, let $C\,(X)$ be the space of all the complex continuous functions on X. If the involution is given by the usual complex conjugation for elements on $\mathbb{C}$, and the norm is the so called sup-norm:

$$\| f \|_\infty \equiv \sup_{x \in X} |f\,(x)| \tag{A.8}$$

then $C\,(X)$ is a unital C^*-algebra. If the space X is locally compact (each point has a compact neighborhood), then $C_0\,(X)$ is the space of complex continuous functions vanishing at infinity (this means that for each $\epsilon > 0$ there is a compact subset $K \subset X$ such that $|f\,(x)| < \epsilon$ for all x outside K). Equipped with the usual involution, and the sup-norm, the space $C_0\,(X)$ is a nonunital C^*-algebra.

The commutative GNS theorem proves that these two examples are paradigmatic. For every commutative C^*-algebra $\mathcal{A}$, there exists a locally compact space X such that $\mathcal{A}$ is isometrically isomorphic to the C^*-algebra $C_0\,(X)$. Here isometrically means that this isomorphism preserves the norm of the elements of the algebras. If the $\mathcal{A}$ is unital, then X can be proved to be compact. Moreover, X is proved to be homeomorphic to the set of pure states of the C^*-algebra $\mathcal{A}$. One could say that any commutative C^*-algebra can be realised

[2] This result is not valid for a Banach algebra, where the condition that the norm of the identity element is 1 must be put as a definition.

as the C^*-algebra of complex valued functions over a locally compact Hausdorff space.

The space $\mathcal{B}(\mathcal{H})$ of bounded operators on a complex separable Hilbert space, with the involution given by the adjoint operation, and the norm given by A.2, is a unital C^*-algebra. The GNS theorem proves that, for every noncommutative C^*-algebra $\mathcal{A}$, there exists an Hilbert space $\mathcal{H}$ such that $\mathcal{A}$ is isomorphic to a norm closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. The constructive GNS proof of this theorem proceeds defining, starting from a state $\sigma \in \mathcal{A}$, a separable Hilbert space $\mathcal{H}_\sigma$, and a representation (a $*$-morphism):

$$\pi_\sigma \, : \mathcal{A} \mapsto \mathcal{B}(\mathcal{H}_\sigma)$$

This representation is cyclic: this means that in $\mathcal{H}_\sigma$ there is a cyclic vector ψ_σ, that is a vector such that the span of $\pi_\sigma(A) \cdot \psi_\sigma$ for all $A \in \mathcal{A}$, coincides with the whole $\mathcal{H}_\sigma$. With GNS representations $(\pi_\sigma, \mathcal{H}_\sigma)$ it is possible to introduce a universal representation $\pi_{\mathcal{U}}$, given by their direct sum, on the space:

$$\mathcal{H}_{\mathcal{U}} \equiv \oplus_{\sigma \in \mathcal{S}(\mathcal{A})} \mathcal{H}_\sigma \tag{A.9}$$

It can be proved that the universal representation $(\pi_U, \mathcal{H}_U)$ defines a C^*-algebra isomorphism between $\mathcal{A}$ and $\mathcal{B}(\mathcal{H}_{\mathcal{U}})$. In this approach, a state $\sigma \in \mathcal{S}(\mathcal{A})$ is a pure state if and only if its associated GNS representation $(\pi_\sigma, \mathcal{H}_\sigma)$ is irreducible. It is then represented as a ray of the vector space $\mathcal{H}_\sigma$.

A.2 Fourier symplectic transform

In the standard approach, Fourier analysis starts considering an integrable function on a vector space, equipped with the translationally invariant Lebesgue measure, $f \in \mathcal{L}^1(\mathbb{R}^m, dz)$, so that Fourier transform of f is defined by:

$$\check{f}(w) = \int \frac{dz}{(2\pi)^{m/2}} f(z)\, e^{-iw\cdot z} \tag{A.10}$$

This map can be inverted, modulo a measurability condition for the the local variation of the function f: the inverse, the Fourier antitransform, is given by:

$$f(z) = \int \frac{dw}{(2\pi)^{m/2}} \check{f}(w)\, e^{iz\cdot w} \tag{A.11}$$

Plancherel theorem shows the way this map can be extended to the space of square integrable functions, and that, on this Hilbert space, it defines a unitary operator. It is usually said that the two variables, in this example z and w, are Fourier conjugate. From a more geometrical point of view, this map is defined via a unitary representation of the translation group $(\mathbb{R}^m, +)$. The integral kernel in the expressions above is a phase factor, that defines a representation of the unitary action of the linear functional w on the vector z. This kind of harmonic analysis perfectly fits with the general theory of noncompact abelian groups: the dual space of such groups coincides with the group itself. So, $(\mathbb{R}^m)^* = \mathbb{R}^m$. The action of the w coordinate on the z coordinate is written in terms of a scalar product, and the expression $z \cdot w$ can be seen as the image of a symmetric 2-form on the pair of vectors z and w, following the natural identification.

In the setting described in the text, there is actually a vector space, equipped with a symplectic, translationally invariant, 2-form, $\left(L \simeq \mathbb{R}^{2n}, \omega\right)$. It seems natural to perform an harmonic analysis for this homogeneous space using this very structure. Such a symplectic form can always be brought in the canonical, Darboux, form, represented by a matrix $\tilde{\omega}^3$, via an invertible transformation. Moreover, such a T is not uniquely determined by ω. The composition of T with an arbitrary symplectic tranformation gives another transformation, still reducing ω in the canonical form:

$$T^t \tilde{\omega} T = \omega \tag{A.12}$$

The symplectic Fourier transform [16] is ($|T|$ is the determinant of T):

$$\tilde{f}(w) = \int \frac{dz}{(2\pi)^n} |T| \, f(z) \, e^{-i\omega(z,w)} \tag{A.13}$$

This can be inverted:

$$f(z) = \int \frac{dw}{(2\pi)^n} |T| \, \tilde{f}(w) \, e^{i\omega(z,w)} \tag{A.14}$$

In this definition it is encoded the ambiguity in the realization of T. This definition can be seen to be covariant for symplectic transformation. If Φ is a symplectic transformation in $\mathbb{R}^{2n}$, then it induces a transformation (the push-forward) in the set of functions on that space:

$$\Phi_* f = f \circ \Phi^{-1} \tag{A.15}$$

or equivalently:

$$\widetilde{(\Phi_* f)} = \Phi_* \left(\tilde{f}\right) \tag{A.16}$$

A.3 Generalised coherent states

In the main text the concept of generalised coherent states has been extensively used. It has been used to describe the introduction of a set of quantizer operators in chapter one, and to define, following the work of Berezin on quantization, maps from operators to functions (symbols) on the sphere and on the plane in chapter three. The aim of this appendix is to briefly recollect the main definitions and results, to make easier the reading of the main text. The main reference is [31].

The analysis starts considering a Lie group G, and $\hat{U}(g)$ a unitary irreducible representation of this group on a Hilbert space $\mathcal{H}$. Chosen a *fiducial* vector $|\,\psi_0\rangle$ in $\mathcal{H}$, one obtains a set of vectors for each element of the group, acting on it with $\hat{U}(g)$:

$$|\,\psi_g\rangle \equiv \hat{U}(g)\,|\,\psi_0\rangle \tag{A.17}$$

[3] In the usual identification of $\mathbb{R}^{2n}$ with the symplectic phase space of certain classical dynamics $T^*\mathbb{R}^n$, with coordinates (q^a, p_b), the matrix representing the canonical symplectic form is $\tilde{\omega} = dq^a \wedge dp_a$.

Two such vectors are considered *equivalent* if they correspond, quantum-mechanically, to the same state, i.e. if they differ by a phase. So $\mid \psi_g\rangle \simeq \mid \psi_{g'}\rangle$ if $\mid \psi_g\rangle = e^{i\phi(g,g')} \mid \psi_{g'}\rangle$. This condition is equivalent to $\hat{U}\left(g'^{-1}g\right) \mid \psi_0\rangle = e^{i\phi(g,g')} \mid \psi_0\rangle$. If H is the subgroup of G whose elements are represented, by $\hat{U}$, as operators whose action on the fiducial vector is just a multiplication by a phase, then the equivalence relation is among points of G, and the quotient is the space G/H. If H is maximal, then it is called isotropy subgroup for the state $\mid \psi_0\rangle$. Choosing a representative $g\left(x\right)$ in each equivalence class $x \in X = G/H$ (which is a cross section of the fiber bundle G with base X) defines a set of vectors on $\mathcal{H}$, depending, clearly, on G and $\mid \psi_0\rangle$. This set of states is called a *system of coherent states* for G. As it has been presented in section 3.1.2, the state corresponding to the vector $\mid x\rangle$ may be considered as the range of a rank one projector in $\mathcal{H}$. Thus, the system of generalised coherent states determines a set of one dimensional subspaces in $\mathcal{H}$, parametrised by points of the homogeneous space $X = G/H$. An evolution of this analysis drives naturally to the issue of overcompleteness for the system of coherent states, mentioned in (1.124), (3.60), (3.19).

Appendix B

B.1 Product among symbols in the Weyl-Wigner isomorphism

In this appendix there are recollected the calculations performed following the definition of the Weyl-Wigner isomorphism between a subset of functions in $\mathcal{F}\left(G \times \tilde{G}\right)$ (G is a compact simple Lie group, and $\tilde{G}$ is a discrete set, whose values label the UIR's of the group, and the elements of the basis chosen in each space for each *inequivalent* representation) and the set of Hilbert-Schmidt operators on the space $\mathcal{H} = \mathcal{L}^2\left(G, d\mu\right)$.

In this Hilbert space, one can consider a set of generalized states that constitute a basis, such that, if $\psi \in \mathcal{H}$, then:

$$\langle g \mid \psi \rangle = \psi\left(g\right)$$

and satisfy an uncountable form of completeness relation:

$$\int d\mu \ \mid g\rangle\langle g \mid = 1 \quad \langle g \mid g' \rangle \ = \ \delta\left(g' g^{-1}\right)$$

Two sets of operators are defined in such a way that:

$$\left(\hat{V}\left(g'\right)\psi\right)\left(g\right) = \psi\left(g'^{-1} g\right) \tag{B.1}$$

$$\left(\hat{U}\left(jmn\right)\psi\right)\left(g\right) = D^j_{mn}\left(g\right)\psi\left(g\right) \tag{B.2}$$

where $D^j_{mn}\left(g\right)$ are the matrix elements of the representative of group element g in the representation labelled by j.

Quantizer operators are (2.76):

$$\hat{W}\left(g, jmn\right) = \sum_{j',m',n'} N_{j'} \int d\mu' \hat{U}\left(j'n'm'\right)\hat{V}\left(g'\right) D^j_{mn}\left(g'\right) D^{j'}_{m'n'}\left(g^{-1} s_0\left(g'^{-1}\right)\right) \tag{B.3}$$

then the isomorphism is realized by:

$$\hat{A} = \sum_{j,m,n} N_j \int_G d\mu \, A\left(g, jmn\right) \hat{W}\left(g, jnm\right) \tag{B.4}$$

$$A\left(g, jmn\right) = Tr\, \hat{A}\hat{W}\left(g, jmn\right) \tag{B.5}$$

The product among symbols is (eq.2.84):

$$(A * B)(g,\gamma) = Tr\hat{A}\hat{B}\hat{W}(g,\gamma) \tag{B.6}$$

$$(A * B)(g,\gamma) = \sum_{\tilde{\gamma}\,\check{\gamma}} N_{\tilde{j}} N_{\check{j}} \int_G d\tilde{\mu} \int_G d\check{\mu}\, A(\tilde{g},\tilde{\gamma})\, B(\check{g},\check{\gamma}) \left[Tr\,\hat{W}(g,\gamma)\,\hat{W}(\tilde{g},\underline{\tilde{\gamma}})\,\hat{W}(\check{g},\underline{\check{\gamma}})\right] \tag{B.7}$$

As it has been stressed in the main text, this product is non local, and the integral kernel is given by the trace term between square brackets. To analyse this term, the first step is to study the possibility of a kind of inversion of (2.76). From the fact that D functions are matrix elements of a UIR, one has:

$$\begin{aligned} D^{j'}_{m'n'}\left(g^{-1}s_0\left(g'^{-1}\right)\right) &= \sum_{s=1}^{N_{j'}} D^{j'}_{m's}\left(g^{-1}\right) D^{j'}_{sn'}\left(s_0\left(g'^{-1}\right)\right) \\ &= \sum_{s=1}^{N_{j'}} \left(D^{j'}_{sm'}(g)\right)^* D^{j'}_{sn'}\left(s_0\left(g'^{-1}\right)\right) \end{aligned} \tag{B.8}$$

and then

$$\begin{aligned} \hat{W}(g,\gamma) = \sum_{\gamma'} N_{j'} \int_G dg' \quad & \hat{U}(j',n',m')\,\hat{V}(g')\,\cdot \\ & \cdot D^{j}_{mn}(g') \sum_{s=1}^{N_{j'}} \left(D^{j'}_{sm'}(g)\right)^* D^{j'}_{sn'}\left(s_0\left(g'^{-1}\right)\right) \end{aligned}$$

Since D functions are an orthonormal basis for $\mathcal{H}$, one can put:

$$\begin{aligned} \int_G dg\,\hat{W}(g,\gamma)\,D^{j''}_{n''m''}(g) &= \sum_{\gamma'} \int_G dg' \int_G dg\,\hat{U}(j',n',m')\,\hat{V}(g')\,D^{j}_{mn}(g')\,\cdot \\ &\cdot \sum_{s=1}^{N_{j'}} \left(D^{j'}_{sm'}(g)\right)^* N_{j'} D^{j''}_{n''m''}(g)\,D^{j'}_{sn'}\left(s_0\left(g'^{-1}\right)\right) \end{aligned}$$

Integration on dg in the RHS gives:

$$\int_G dg\,\hat{W}(g,\gamma)\,D^{j''}_{n''m''}(g) = \sum_{n'} \int_G dg'\hat{U}(j'',n',m'')\,\hat{V}(g')\,D^{j}_{mn}(g')\,D^{j''}_{n''n'}\left(s_0\left(g'^{-1}\right)\right) \tag{B.9}$$

Again using orthonormality of D functions:

$$\begin{aligned} \sum_{\gamma} \int_G dg\,\hat{W}(g,\gamma) \quad &\cdot \quad D^{j''}_{n''m''}(g)\left(D^{j}_{mn}(\tilde{g})\right)^* N_j = \\ &\cdot \quad \sum_{n'}\sum_{\gamma} \int_G dg'\hat{U}(j'',n',m'')\,\hat{V}(g')\,\cdot \\ &\cdot \quad D^{j}_{mn}(g')\left(D^{j}_{mn}(\tilde{g})\right)^* N_j D^{j''}_{n''n'}\left(s_0\left(g'^{-1}\right)\right) \end{aligned} \tag{B.10}$$

Summation over discrete indices γ gives a $\delta\left(\tilde{g}g'^{-1}\right)$ factor in RHS:

$$\begin{aligned} \sum_{\gamma} \int_G dg\,\hat{W}(g,\gamma) \quad &\cdot \quad D^{j''}_{n''m''}(g)\left(D^{j}_{mn}(\tilde{g})\right)^* N_j = \\ &= \sum_{n'} \hat{U}(j'',n',m'')\,\hat{V}(\tilde{g})\,D^{j''}_{n''n'}\left(s_0\left(\tilde{g}^{-1}\right)\right) \end{aligned} \tag{B.11}$$

Now one can even saturate the index n'':

$$\begin{aligned}
\sum_{n''}\sum_{\gamma}\int_G dg\,\hat{W}(g,\gamma) \quad &\cdot\quad D^{j''}_{n''m''}(g)\left(D^{j}_{mn}(\tilde{g})\right)^{*} N_j D^{j''}_{kn''}(s_0(\tilde{g})) = \\
&= \sum_{n'}\hat{U}(j'',n',m'')\,\hat{V}(\tilde{g})\,D^{j''}_{kn'}(e) = \\
&= \hat{U}(j'',k,m'')\,\hat{V}(\tilde{g}) \qquad \text{(B.12)}
\end{aligned}$$

the last equality comes from $D^{j''}_{kn'}(e)=\delta_{kn'}$ since e is the identity of the group G. Finally one has:

$$\hat{U}\left(\tilde{j},\tilde{n},\tilde{m}\right)\hat{V}(\tilde{g}) = \sum_{\gamma}\int_G dg\,\hat{W}(g,\gamma)\,D^{\tilde{j}}_{\tilde{n}\tilde{m}}(s_0(\tilde{g})\,g)\left(D^{j}_{mn}(\tilde{g})\right)^{*} N_j \qquad \text{(B.13)}$$

This can be seen as a sort of antitransform of (2.76).

The second step of the analysis just gives the composition properties of $\hat{U}$and $\hat{V}$ operators:

$$\hat{U}(j',n',m')\,\hat{U}(j'',n'',m'') = \sum_{JNM,\lambda} C^{j',j'',J\,\lambda}_{n'm',n''m'',NM}\hat{U}(J,N,M) \qquad \text{(B.14)}$$

$$\hat{U}(j',n',m')\,\hat{V}(g')\,\hat{U}(j'',n'',m'')\,\hat{V}(g'') = \sum_{k=1}^{N_{j''}}\sum_{JNM,\lambda} D^{j''}_{n''k}\left(g'^{-1}\right) C^{j',j'',J\,\lambda}_{n'm',km'',NM}\hat{U}(J,N,M)\,\hat{V}(g'g'') \qquad \text{(B.15)}$$

where the meaning of the coefficient has been explained in the main text.

The third step is to study the composition properties in the set of $\hat{W}$. From the last relations, it is easy to see that:

$$\begin{aligned}
\hat{W}(g,\gamma)\,\hat{W}(\tilde{g},\tilde{\gamma}) \quad &= \quad \sum_{\gamma'\,\gamma''} N_{j'}N_{j''}\int_G dg'\int_G dg''\sum_{k=1}^{N_{j''}}\sum_{JNM,\lambda}\cdot \\
&\cdot \quad D^{j''}_{n''k}\left(g'^{-1}\right) C^{j',j'',J\,\lambda}_{n'm',km'',NM}\hat{U}(J,N,M)\,\hat{V}(g'g'')\,\cdot \\
&\cdot \quad D^{j}_{mn}(g')\,D^{j'}_{m'n'}\left(g^{-1}s_o\left(g'^{-1}\right)\right) D^{\tilde{j}}_{\tilde{m}\tilde{n}}(g'')\,D^{j''}_{m''n''}\left(\tilde{g}^{-1}s_o\left(g''^{-1}\right)\right)
\end{aligned}$$

Substitution of (B.13) into the RHS of the previous relation gives:

$$\begin{aligned}
\hat{W}(g,\gamma)\,\hat{W}(\tilde{g},\tilde{\gamma}) \quad &= \quad \sum_{\gamma'\,\gamma''\,\gamma'''} N_{j'}N_{j''}N_{j'''}\int_G dg'\int_G dg''\int_G dg'''\sum_{k=1}^{N_{j''}}\sum_{JNM,\lambda}\cdot D^{j''}_{n''k}\left(g'^{-1}\right) \\
&\cdot \quad C^{j',j'',J\,\lambda}_{n'm',km'',NM}\hat{W}(g''',\gamma''')\,D^{J}_{NM}(s_o(g'g'')\,g''')\left(D^{j'''}_{m'''n'''}(g'g'')\right)^{*}\cdot \\
&\cdot \quad D^{j}_{mn}(g')\,D^{j'}_{m'n'}\left(g^{-1}s_o\left(g'^{-1}\right)\right)\cdot \\
&\cdot \quad D^{\tilde{j}}_{\tilde{m}\tilde{n}}(g'')\,D^{j''}_{m''n''}\left(\tilde{g}^{-1}s_o\left(g''^{-1}\right)\right) \qquad \text{(B.16)}
\end{aligned}$$

The definition (2.84) indicates that the problem is evaluating the trace of the product of three $\hat{W}$ operators. One has:

$$\begin{aligned} Tr\left[\hat{W}(g,\gamma)\,\hat{W}(\tilde{g},\tilde{\gamma})\,\hat{W}(\breve{g},\breve{\gamma})\right] &= \sum_{\gamma'\,\gamma''\,\gamma'''} N_{j'}N_{j''}N_{j'''}\int_G dg'\int_G dg''\int_G dg''' \sum_{k=1}^{N_{j''}}\sum_{JNM,\lambda} \cdot \\ &\cdot\; D^{j''}_{n''k}\left(g'^{-1}\right) C^{j',j'',J\,\lambda}_{n'm',km'',NM} D^{J}_{NM}\left(s_o\left(g'g''\right)g'''\right) \cdot \\ &\cdot\; \left(D^{j'''}_{m'''n'''}\left(g'g''\right)\right)^* D^{(j}_{mn}\left(g'\right) D^{j'}_{m'n'}\left(g^{-1}s_o\left(g'^{-1}\right)\right) \cdot \\ &\cdot\; D^{\tilde{j}}_{\tilde{m}\tilde{n}}\left(g''\right) D^{j''}_{m''n''}\left(\tilde{g}^{-1}s_o\left(g''^{-1}\right)\right) \cdot \\ &\cdot\; Tr\left[\hat{W}\left(g''',\gamma'''\right)\hat{W}\left(\breve{g},\breve{\gamma}\right)\right] \end{aligned} \tag{B.17}$$

Since:

$$Tr\left[\hat{W}\left(g''',\gamma'''\right)\hat{W}\left(\breve{g},\breve{\gamma}\right)\right] = \frac{1}{N_{j'''}}\delta_{j'''\breve{j}}\delta_{m'''\breve{n}}\delta_{n'''\breve{m}}\delta\left(\breve{g}^{-1}g'''\right) \tag{B.18}$$

So one obtains the final expression for the integral kernel of the star product (2.84):

$$\begin{aligned} Tr\left[\hat{W}(g,\gamma)\,\hat{W}\left(\tilde{g},\underline{\tilde{\gamma}}\right)\hat{W}\left(\breve{g},\underline{\breve{\gamma}}\right)\right] &= \sum_{\gamma'\,\gamma''}\sum_{\Gamma,\lambda}\int_G dg'\int_G dg'' \sum_{k=1}^{N_{j''}} N_{j'}N_{j''} D^{j''}_{n''k}\left(g'^{-1}\right) \cdot \\ &\cdot\; C^{j',j'',J\,\lambda}_{n'm',km'',NM} D^{J}_{NM}\left(s_o\left(g'g''\right)\breve{g}\right)\left(D^{\breve{j}}_{\breve{m}\breve{n}}\left(g'g''\right)\right)^* \cdot \\ &\cdot\; D^{j}_{mn}\left(g'\right) D^{j'}_{m'n'}\left(g^{-1}s_o\left(g'^{-1}\right)\right) D^{\tilde{j}}_{\tilde{n}\tilde{m}}\left(g''\right) \cdot \\ &\cdot\; D^{j''}_{m''n''}\left(\tilde{g}^{-1}s_o\left(g''^{-1}\right)\right) \end{aligned} \tag{B.19}$$

(here Γ is a short for (J,N,M))

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